Chapter: Calculus (Higher Order Derivatives, Monotonicity, Maxima/Minima)

General Instructions

1. Total Questions: **20**
2. Duration: **60 Minutes**
3. All questions are compulsory.
4. Read each question carefully before answering.
5. Choose the most appropriate answer from the given options.
6. Use of calculator or electronic devices is strictly prohibited.

1. If $f(x) = (x-1)(x-2)(x-3)$, show that there exist at least two values of c in the interval $(1, 3)$ such that $f'(c) = 0$ using Rolle's Theorem.

2. If $y = (x + \sqrt{x^2+1})^m$, prove that $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$.

3. Find the interval in which the function $f(x) = \frac{4x^2+1}{x}$ is strictly decreasing.

4. Determine the values of a for which the function $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ is strictly decreasing for all $x \in \mathbb{R}$.

5. Let $f(x)$ be a polynomial of degree 4 such that $f'(0) = 0$ and $f(x)$ has local extrema at $x = 1, 2, 3$. If $f(0) = 0$, find the expression for $f(x)$.

6. Find the maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$.

7. If $f(x) = x^3 + px^2 + qx + r$, find the condition for which $f(x)$ is always increasing.

8. A wire of length 28m is to be cut into two pieces. One piece is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area is minimum?

9. Find the point on the curve $y^2 = 4x$ which is nearest to the point $(2, 1)$.

10. If $f(x) = \frac{\sin x + \cos x}{e^x}$, find the value of $f''(\pi/2)$.

11. Prove that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

12. Find the points of inflection of the curve $f(x) = e^{-x^2}$.

13. Let $f(x)$ be a function such that $f(x) = |x|^3$. Prove that $f''(x)$ exists for all real x and find it.

14. Find the values of x for which the function $f(x) = x - \log(1+x)$ is increasing.

15. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ has a local maximum at $x = p$ and local minimum at $x = q$ such that $p^2 = q$, find the value of a (given $a > 0$).

16. Determine the maximum area of a rectangle that can be inscribed in a circle of radius r.

17. Show that for $x > 0$, the inequality $x - \frac{x^2}{2} < \log(1+x) < x$ holds.

18. If $x = \sin t, y = \sin pt$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

19. Find the local maximum and minimum values of $f(x) = \sin 2x - x$ in $[0, \pi]$.

20. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Solutions

1. $f(x)$ is a polynomial, continuous on $[1,3]$ and differentiable on $(1,3)$. $f(1)=f(2)=0$ and $f(2)=f(3)=0$. By Rolle's Theorem, there is $c_1 \in (1,2)$ s.t. $f'(c_1)=0$ and $c_2 \in (2,3)$ s.t. $f'(c_2)=0$.

2. $y_1 = m(x+\sqrt{x^2+1})^{m-1}(1+\frac{x}{\sqrt{x^2+1}}) = \frac{my}{\sqrt{x^2+1}}$. Squaring: $(x^2+1)y_1^2 = m^2y^2$. Differentiating again: $(x^2+1)2y_1y_2 + 2xy_1^2 = 2m^2yy_1$. Divide by $2y_1$ to get the result.

3. $f'(x) = 4 - \frac{1}{x^2}$. For strictly decreasing, $4 - \frac{1}{x^2} < 0 \Rightarrow 4 < \frac{1}{x^2} \Rightarrow x^2 < 1/4 \Rightarrow x \in (-1/2, 0) \cup (0, 1/2)$.

4. $f'(x) = 3(a+2)x^2 - 6ax + 9a$. For strictly decreasing, $f'(x) < 0$ for all x. This requires $a+2<0$ and Discriminant $D<0$. $36a^2 - 4(3(a+2))(9a) < 0 \Rightarrow a^2 - a(a+2) < 0 \Rightarrow -2a < 0 \Rightarrow a > 0$. But $a < -2$. No such a exists.

5. $f'(x)$ has roots at 1, 2, 3 and is of degree 3. $f'(x) = k(x-1)(x-2)(x-3)$. Integrating: $f(x) = k(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x) + C$. $f(0)=0 \Rightarrow C=0$.

6. Slope $m = y' = -3x^2 + 6x + 9$. To maximize m, $m' = -6x + 6 = 0 \Rightarrow x = 1$. Max slope $m(1) = -3+6+9 = 12$.

7. $f'(x) = 3x^2 + 2px + q$. For always increasing, $f'(x) \geq 0 \forall x$. This requires $D \leq 0 \Rightarrow (2p)^2 - 4(3)(q) \leq 0 \Rightarrow p^2 \leq 3q$.

8. Let x be length for square ($side = x/4$) and $28-x$ for circle ($radius = \frac{28-x}{2\pi}$). Area $A = \frac{x^2}{16} + \frac{(28-x)^2}{4\pi}$. $dA/dx = 0 \Rightarrow x = \frac{112}{4+\pi}$.

9. Minimize $D^2 = (x-2)^2 + (y-1)^2$ where $x = y^2/4$. $g(y) = (\frac{y^2}{4} - 2)^2 + (y-1)^2$. $g'(y) = 0$ gives $y = 2$. Point is $(1,2)$.

10. $f'(x) = \frac{e^x(\cos x - \sin x) - e^x(\sin x + \cos x)}{e^{2x}} = \frac{-2\sin x}{e^x}$. $f''(x) = \frac{-2e^x \cos x + 2e^x \sin x}{e^{2x}}$. $f''(\pi/2) = \frac{2}{e^{\pi/2}}$.

11. $V = \frac{1}{3}\pi(l\sin\alpha)^2(l\cos\alpha) = \frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha$. $dV/d\alpha = 0 \Rightarrow \sin\alpha(2\cos^2\alpha - \sin^2\alpha) = 0 \Rightarrow \tan^2\alpha = 2 \Rightarrow \alpha = \tan^{-1}\sqrt{2}$.

12. $f'(x) = -2xe^{-x^2}$. $f''(x) = (4x^2-2)e^{-x^2}$. $f''(x) = 0 \Rightarrow 4x^2 = 2 \Rightarrow x = \pm 1/\sqrt{2}$.

13. $f(x) = x^3$ for $x \geq 0$ and $-x^3$ for $x < 0$. $f'(x) = 3x^2$ for $x \geq 0$ and $-3x^2$ for $x < 0$. $f''(x) = 6x$ for $x \geq 0$ and $-6x$ for $x < 0$. $f''(x) = 6|x|$.

14. $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$. Increasing when $x/(1+x) > 0$. For $x > -1$, this means $x > 0$. **Answer:** $(0, \infty)$.

15. $f'(x) = 6x^2 - 18ax + 12a^2 = 6(x-a)(x-2a)$. Roots $a, 2a$. Since $a > 0$, max at $p = a$, min at $q = 2a$. $a^2 = 2a \Rightarrow a = 2$ (since $a > 0$).

16. Let sides be $2r\cos\theta, 2r\sin\theta$. Area $A = 4r^2\sin\theta\cos\theta = 2r^2\sin 2\theta$. Max area is $2r^2$ when $\theta = \pi/4$ (a square).

17. Let $g(x) = x - \log(1+x)$. $g'(x) = x/(1+x) > 0$ for $x > 0 \Rightarrow \log(1+x) < x$. Let $h(x) = \log(1+x) - (x - x^2/2)$. $h'(x) = \frac{x^2}{1+x} > 0$ for $x > 0 \Rightarrow \log(1+x) > x - x^2/2$.

18. Standard differentiation using chain rule and quotient rule; rearrange to form the given differential equation.

19. $f'(x) = 2\cos 2x - 1 = 0 \Rightarrow x = \pi/6, 5\pi/6$. Max value at $x = \pi/6$ is $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$. Min value at $x = 5\pi/6$ is $-\frac{\sqrt{3}}{2} - \frac{5\pi}{6}$.

20. Perimeter $2r + 2h + \pi r = 10$. Area $A = 2rh + \frac{1}{2}\pi r^2$. Substituting h and differentiating gives $r = \frac{10}{\pi+4}$. **Answer: Radius $r = \frac{10}{\pi+4}$ m, Height $h = \frac{10}{\pi+4}$ m.**