Unit III: Calculus - Continuity and Differentiability

SOLUTIONS

Solutions

1. **Solution:** LHL: $\lim_{x\to 0^-} \frac{2\sin^2 2x}{x^2} = 2\cdot 4 = 8$. RHL: $\lim_{x\to 0^+} \frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{(\sqrt{16+\sqrt{x}}-4)(\sqrt{16+\sqrt{x}}+4)} = \lim_{x\to 0^+} \frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{\sqrt{x}} = 8$. For continuity, $a = 8$.

2. **Solution:** For continuity at $x = 1$: $1 + 3 + a = b + 2 \Rightarrow a - b = -2$. For differentiability: $LHD = 2x + 3|_{x=1} = 5$; $RHD = b$. Thus $b = 5$ and $a = 5 - 2 = 3$.

3. **Solution:** $y^2 = \log x + y$. Diff. w.r.t x: $2yy' = \frac{1}{x} + y' \Rightarrow y'(2y-1) = \frac{1}{x}$.

4. **Solution:** $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(h)+xh^2+x^2h}{h} = \lim_{h\to 0}\left(\frac{f(h)}{h} + xh + x^2\right) = 1 + x^2$.

5. **Solution:** $dx/dt = a(-\sin t + \sin t + t\cos t) = at\cos t$. $dy/dt = a(\cos t - \cos t + t\sin t) = at\sin t$. $dy/dx = \tan t$. $d^2y/dx^2 = \sec^2 t \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{1}{at\cos t} = \frac{\sec^3 t}{at}$.

6. **Solution:** Let $x = \tan\theta$. $u = \tan^{-1}(\frac{\sec\theta - 1}{\tan\theta}) = \theta/2$. $v = \sin^{-1}(\frac{2\tan\theta}{1+\tan^2\theta}) = 2\theta$. $du/dv = \frac{1/2}{2} = 1/4$.

7. **Solution:** Take log: $x\log y = y - x$. Diff. w.r.t x: $\log y + \frac{x}{y}y' = y' - 1 \Rightarrow y'(1 - \frac{x}{y}) = 1 + \log y$. Since $x = \frac{y}{1+\log y}$, substituting gives the result.

8. **Solution:** For $x \in (-1, 1)$, $f(x) = (1-x) + (x+1) = 2$. $f'(0) = 0$. At $x = 1$, LHD is 0, RHD is 2 ($f(x) = 2x$ for $x > 1$). Not differentiable at $x = 1$.

9. **Solution:** $y_1 = m(x + \sqrt{x^2+1})^{m-1}(1 + \frac{x}{\sqrt{x^2+1}}) = \frac{my}{\sqrt{x^2+1}}$. Squaring: $(x^2+1)y_1^2 = m^2y^2$. Differentiating again yields the differential equation.

10. **Solution:** Potential discontinuities for $[x^2]$ in $(1, 2)$ are at $x^2 = 2, 3 \Rightarrow x = \sqrt{2}, \sqrt{3}$. However, at these points $\sin(\pi x) \neq 0$, so jump discontinuities remain.

11. **Solution:** $dx/d\theta = \sec\theta\tan\theta + \sin\theta$. $dy/d\theta = n\sec^n\theta\tan\theta + n\cos^{n-1}\theta\sin\theta$. Using $dy/dx = (dy/d\theta)/(dx/d\theta)$ and squaring leads to the result.

12. **Solution:** $f'(x) = \frac{1}{x\log x}$. $f''(x) = -\frac{(1+\log x)}{(x\log x)^2}$. At $x = e$, $f''(e) = -\frac{2}{e^2}$.

13. **Solution:** $f'(x) = f(x)f'(0)$. $f'(5) = f(5)f'(0) = 2\times 3 = 6$.

14. **Solution:** Let $x = \sin\theta$ and $2/\sqrt{13} = \sin\alpha, 3/\sqrt{13} = \cos\alpha$. $y = \cos^{-1}(\sin(\theta - \alpha)) = \cos^{-1}(\cos(\pi/2 - \theta + \alpha)) = \pi/2 - \sin^{-1}x + \alpha$. $dy/dx = -1/\sqrt{1-x^2}$.

15. **Solution:** Continuity: $\lim_{x\to 0} x\sin(1/x) = 0$. Differentiability: $\lim_{h\to 0}\frac{h\sin(1/h)-0}{h} = \lim_{h\to 0}\sin(1/h)$, which does not exist (oscillates).

16. **Solution:** $y_1 = \frac{1}{1+x^2} \Rightarrow (1+x^2)y_1 = 1$. Diff: $(1+x^2)y_2 + 2xy_1 = 0$. Diff again: $(1+x^2)y_3 + 2xy_2 + 2xy_2 + 2y_1 = 0 \Rightarrow (1+x^2)y_3 + 4xy_2 + 2y_1 = 0$.

17. **Solution:** $\lim_{x\to\pi/2}\frac{k\cos x}{\pi - 2x}$. Using L'Hôpital: $\lim_{x\to\pi/2}\frac{-k\sin x}{-2} = k/2$. Set $k/2 = 3 \Rightarrow k = 6$.

18. **Solution:** $e^x + e^y = e^x \cdot e^y \Rightarrow e^{-y} + e^{-x} = 1$. Diff: $-e^{-y}y' - e^{-x} = 0 \Rightarrow y' = -e^{-x}/e^{-y} = -e^{y-x}$.

19. **Solution:** $x = x^3 \Rightarrow x = 0, 1, -1$. Check LHD/RHD at these intersection points. Sharp corners occur at $x = 1$ and $x = -1$.

20. **Solution:** $y_1 = \frac{\cos(\log x)}{x} \Rightarrow xy_1 = \cos(\log x)$. Diff: $xy_2 + y_1 = -\frac{\sin(\log x)}{x}$. $x^2y_2 + xy_1 = -y \Rightarrow x^2y_2 + xy_1 + y = 0$.