SAMPLE QUESTION PAPER - 2025-26
CLASS XII - MATHEMATICS (041)

Time Allowed: 3 Hours **Maximum Marks: 80**

General Instructions:

1. This Question Paper contains **38** questions. All questions are compulsory.
2. The question paper is divided into FIVE Sections – A, B, C, D and E.
3. Section **A** comprises of **20** questions of **1** mark each.
4. Section **B** comprises of **5** questions of **2** marks each.
5. Section **C** comprises of **6** questions of **3** marks each.
6. Section **D** comprises of **4** questions of **5** marks each.
7. Section **E** comprises of **3** Case Study Based Questions of **4** marks each.
8. There is no overall choice in the question paper. However, an internal choice has been provided in **2** questions in Section B, **3** questions in Section C, **2** questions in Section D and **2** questions in Section E (in the sub-parts).
9. Use of calculators is **not** permitted.

SECTION A (20 Marks)

This section comprises **20** *questions of* **1** *mark each. Questions 1-18 are Multiple Choice Questions (MCQs) and questions 19-20 are Assertion-Reason based questions.*

Multiple Choice Questions (MCQs)

1. Let R be a relation on the set $\mathbb{N}$ of natural numbers defined by xRy if x divides y. Then R is:
   (a) Reflexive and Symmetric
   (b) Transitive but not Symmetric
   (c) Reflexive and Transitive
   (d) An Equivalence relation

   Answer: (b) Transitive but not Symmetric

   Solution: For relation R defined as xRy if x divides y:

   - **Reflexive:** For any natural number x, x divides x, so xRx is true. Thus R is reflexive.
   - **Symmetric:** If xRy, then x divides y. But this does not imply y divides x (e.g., 2 divides 4 but 4 does not divide 2). Thus R is not symmetric.
   - **Transitive:** If xRy and yRz, then x divides y and y divides z, which implies x divides z. Thus R is transitive.

   Therefore, R is reflexive and transitive but not symmetric, making option (b) correct.

2. If $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)+\cos^{-1}\left(\frac{1}{2}\right)\right)$ is equal to:
   (a) 0
   (b) 1
   (c) $\frac{1}{2}$
   (d) $\frac{\sqrt{3}}{2}$

Answer: (a) 0

Solution: We know that $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$. Therefore,

$$\begin{aligned}
\cos\left(\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{2}\right)\right) &= \cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) \\
&= \cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) \\
&= \cos\left(\frac{\pi}{6} + \frac{2\pi}{6}\right) \\
&= \cos\left(\frac{3\pi}{6}\right) = \cos\left(\frac{\pi}{2}\right) = 0
\end{aligned}$$

Hence, option (a) is correct.

3. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 2$, then $f^{-1}(x)$ is:
   (a) $3x + 2$
   (b) $\frac{x}{3} - 2$
   (c) $\frac{x+2}{3}$
   (d) $\frac{x-2}{3}$

   Answer: (c) $\frac{x+2}{3}$

   Solution: To find $f^{-1}(x)$, let $y = f(x) = 3x - 2$. Interchanging x and y, we get $x = 3y - 2$. Solving for y:

$$\begin{aligned}
x &= 3y - 2 \\
3y &= x + 2 \\
y &= \frac{x+2}{3}
\end{aligned}$$

   Therefore, $f^{-1}(x) = \frac{x+2}{3}$, which is option (c).

4. The principal value of $\tan^{-1}\left(-\sqrt{3}\right)$ is:
   (a) $\frac{2\pi}{3}$
   (b) $-\frac{\pi}{6}$
   (c) $-\frac{\pi}{3}$
   (d) $\frac{\pi}{3}$

   Answer: (c) $-\frac{\pi}{3}$

   Solution: The principal value range for $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We know that $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$. Since $\tan(-\theta) = -\tan(\theta)$, we have $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$. And $-\frac{\pi}{3}$ lies in the principal value range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$, which is option (c).

5. Let $A = \{1, 2, 3\}$. The number of equivalence relations containing $(1, 2)$ is:
   (a) 1
   (b) 2
   (c) 3
   (d) 4

   Answer: (b) 2

   Solution: An equivalence relation on A must be reflexive, symmetric, and transitive. Since $(1, 2)$ is in the relation, by symmetry $(2, 1)$ must also be in the relation. By reflexivity, $(1, 1)$, $(2, 2)$, and $(3, 3)$ are in the relation. Two possible equivalence relations containing $(1, 2)$ are:

- Partition $\{\{1,2\},\{3\}\}$: This gives pairs $(1,1)$, $(2,2)$, $(3,3)$, $(1,2)$, $(2,1)$.
- Partition $\{\{1,2,3\}\}$: This gives all possible pairs.

No other equivalence relations are possible. Therefore, there are exactly 2 such relations, making option (b) correct.

6. If a matrix A is both symmetric and skew-symmetric, then A is a:

   (a) Diagonal matrix

   (b) Zero matrix

   (c) Square matrix

   (d) Identity matrix

   Answer: (b) Zero matrix

   Solution: For a matrix A:

   - Symmetric means $A^T = A$
   - Skew-symmetric means $A^T = -A$

   If A is both, then $A = A^T = -A$, which gives $A = -A$, so $2A = 0$, hence $A = 0$. Therefore, A must be a zero matrix, which is option (b).

7. If A is a square matrix of order 3 and $|A| = -2$, then $|adj(A)|$ is equal to:

   (a) 4

   (b) -2

   (c) -8

   (d) 2

   Answer: (a) 4

   Solution: For a square matrix of order n, we know that $|adj(A)| = |A|^{n-1}$. Here, $n = 3$ and $|A| = -2$. Therefore, $|adj(A)| = |A|^{3-1} = |A|^2 = (-2)^2 = 4$. Hence, option (a) is correct.

8. If A is an invertible matrix of order 2, then $\det(A^{-1})$ is equal to:

   (a) $\det(A)$

   (b) $\frac{1}{\det(A)}$

   (c) 1

   (d) 0

   Answer: (b) $\frac{1}{\det(A)}$

   Solution: For any invertible matrix A, we have $A \cdot A^{-1} = I$. Taking determinants on both sides: $\det(A \cdot A^{-1}) = \det(I)$ $\det(A) \cdot \det(A^{-1}) = 1$ (since $\det(AB) = \det(A)\det(B)$ and $\det(I) = 1$) Therefore, $\det(A^{-1}) = \frac{1}{\det(A)}$, provided $\det(A) \neq 0$. This is option (b).

9. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the inverse of:

   (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

   (d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Answer: (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If A is the inverse of some matrix B, then $B = A^{-1}$. But A itself is the identity matrix I_2, and $I_2^{-1} = I_2$. Therefore, A is the inverse of itself. Verify: $A \times A = I_2 \times I_2 = I_2$, so indeed $A^{-1} = A$. Thus, option (b) is correct.

10. The value of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$ is:

    (a) $2\sqrt{2}$

    (b) 4

    (c) $\pm 2\sqrt{2}$

    (d) ± 4

    Answer: (c) $\pm 2\sqrt{2}$

    Solution: Compute both determinants: Left side: $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = 3(1) - x(x) = 3 - x^2$ Right side: $\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 2(4) = 3 - 8 = -5$

    Equating: $3 - x^2 = -5$

    $$\begin{aligned} 3 - x^2 &= -5 \\ -x^2 &= -5 - 3 \\ -x^2 &= -8 \\ x^2 &= 8 \\ x &= \pm\sqrt{8} = \pm 2\sqrt{2} \end{aligned}$$

    Therefore, option (c) is correct.

11. The value of $\int_{-1}^{1} (x^3 + x)\,dx$ is:

    (a) 2

    (b) 1

    (c) -1

    (d) 0

    Answer: (d) 0

    Solution: Method 1: Direct integration

    $$\begin{aligned} \int_{-1}^{1} (x^3 + x)\,dx &= \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_{-1}^{1} \\ &= \left(\frac{1}{4} + \frac{1}{2}\right) - \left(\frac{1}{4} + \frac{1}{2}\right) \\ &= \left(\frac{1}{4} + \frac{2}{4}\right) - \left(\frac{1}{4} + \frac{2}{4}\right) \\ &= \frac{3}{4} - \frac{3}{4} = 0 \end{aligned}$$

    Method 2: Using property of odd functions $f(x) = x^3 + x$ is an odd function because $f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x)$. For an odd function integrated over symmetric limits $[-a, a]$, the integral is 0. Therefore, $\int_{-1}^{1} (x^3 + x)\,dx = 0$, which is option (d).

12. If $y = \log(\sqrt{\sin x})$, then $\frac{dy}{dx}$ is:

(a) $\cot x$

(b) $\frac{1}{2}\cot x$

(c) $\frac{1}{\sqrt{\sin x}}$

(d) $\frac{\cos x}{\sqrt{\sin x}}$

Answer: (b) $\frac{1}{2}\cot x$

Solution: $y = \log(\sqrt{\sin x}) = \log((\sin x)^{\frac{1}{2}}) = \frac{1}{2}\log(\sin x)$

Differentiating with respect to x:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} \cdot \frac{d}{dx}\log(\sin x) \\ &= \frac{1}{2} \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) \\ &= \frac{1}{2} \cdot \frac{1}{\sin x} \cdot \cos x \\ &= \frac{1}{2} \cdot \frac{\cos x}{\sin x} \\ &= \frac{1}{2}\cot x \end{aligned}$$

Therefore, option (b) is correct.

13. The integrating factor of the differential equation $x\frac{dy}{dx} - y = 2x^2$ is:

(a) x

(b) $-x$

(c) $\frac{1}{x}$

(d) $-\frac{1}{x}$

Answer: (c) $\frac{1}{x}$

Solution: The given differential equation is: $x\frac{dy}{dx} - y = 2x^2$

Dividing throughout by x (for $x \neq 0$): $\frac{dy}{dx} - \frac{y}{x} = 2x$

This is a linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where: $P(x) = -\frac{1}{x}$ and $Q(x) = 2x$

The integrating factor (I.F.) is given by $e^{\int P(x)dx}$:

$$\begin{aligned} \text{I.F.} &= e^{\int -\frac{1}{x}dx} \\ &= e^{-\log x} \\ &= e^{\log(x^{-1})} \\ &= \frac{1}{x} \end{aligned}$$

Therefore, the integrating factor is $\frac{1}{x}$, which is option (c).

14. The slope of the tangent to the curve $y = x^2 - 2x + 1$ at $x = 1$ is:

(a) 0

(b) 1

(c) 2

(d) -1

Answer: (a) 0

Solution: The slope of the tangent to a curve $y = f(x)$ at a point is given by $\frac{dy}{dx}$ at that point.

Given $y = x^2 - 2x + 1$, we differentiate with respect to x: $\frac{dy}{dx} = 2x - 2$

At $x = 1$: $\frac{dy}{dx}\Big|_{x=1} = 2(1) - 2 = 2 - 2 = 0$

Therefore, the slope of the tangent at $x = 1$ is 0, which is option (a).

15. The order and degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^2 + y = 0$ are:

    (a) Order 3, Degree 2

    (b) Order 3, Degree is not defined

    (c) Order 2, Degree 3

    (d) Order 1, Degree 2

    Answer: (a) Order 3, Degree 2

    Solution: The given differential equation is: $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^2 + y = 0$

    - **Order:** The highest order derivative present in the equation is $\frac{d^3y}{dx^3}$, which is of order 3. Therefore, the order of the differential equation is 3.
    - **Degree:** The equation is a polynomial in derivatives. The highest order derivative $\frac{d^3y}{dx^3}$ is raised to the power 2. Therefore, the degree is 2.

    Hence, the order is 3 and degree is 2, making option (a) correct.

16. The minimum value of the function $f(x) = x \log x$ is:

    (a) $-\frac{1}{e}$

    (b) e

    (c) 1

    (d) 0

    Answer: (a) $-\frac{1}{e}$

    Solution: Domain: $x > 0$ for $\log x$ to be defined.

    Differentiating $f(x) = x \log x$ with respect to x: $f'(x) = 1 \cdot \log x + x \cdot \frac{1}{x} = \log x + 1$

    For critical points, set $f'(x) = 0$: $\log x + 1 = 0 \implies \log x = -1 \implies x = e^{-1} = \frac{1}{e}$

    Second derivative test: $f''(x) = \frac{1}{x}$ At $x = \frac{1}{e}$, $f''\left(\frac{1}{e}\right) = \frac{1}{1/e} = e > 0$, indicating a minimum.

    Minimum value: $f\left(\frac{1}{e}\right) = \frac{1}{e}\log\left(\frac{1}{e}\right) = \frac{1}{e} \cdot (-1) = -\frac{1}{e}$

    Therefore, the minimum value is $-\frac{1}{e}$, which is option (a).

17. The projection of the vector $\mathbf{a} = 2\hat{i} - \hat{j} + \hat{k}$ on the vector $\mathbf{b} = \hat{i} + 2\hat{j} + 2\hat{k}$ is:

    (a) $\frac{2}{3}$

    (b) $\frac{1}{3}$

    (c) $\frac{2}{\sqrt{9}}$

    (d) $\frac{1}{9}$

    Answer: (a) $\frac{2}{3}$

    Solution: The projection of vector **a** on vector **b** is given by: Projection $= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

    First, compute $\mathbf{a} \cdot \mathbf{b}$: $\mathbf{a} \cdot \mathbf{b} = (2)(1) + (-1)(2) + (1)(2) = 2 - 2 + 2 = 2$

    Next, compute $|\mathbf{b}|$: $|\mathbf{b}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$

Therefore, Projection $= \frac{2}{3}$, which is option (a).

18. The length of the perpendicular from the origin to the plane $\mathbf{r} \cdot (3\hat{\imath} - 4\hat{\jmath} + 12\hat{k}) = 52$ is:

    (a) 4 units

    (b) 52 units

    (c) 13 units

    (d) 12 units

    Answer: (a) 4 units

    Solution: The equation of the plane is in the form $\mathbf{r} \cdot \mathbf{n} = d$, where $\mathbf{n} = 3\hat{\imath} - 4\hat{\jmath} + 12\hat{k}$ and $d = 52$.

    The perpendicular distance from the origin $(0, 0, 0)$ to the plane is given by: $p = \frac{|d|}{|\mathbf{n}|}$

    Compute $|\mathbf{n}|$: $|\mathbf{n}| = \sqrt{3^2 + (-4)^2 + 12^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$

    Therefore, $p = \frac{52}{13} = 4$ units, which is option (a).

19. If $P(A) = 0.8$ and $P(B|A) = 0.5$, then $P(A \cap B)$ is:

    (a) 0.4

    (b) 0.3

    (c) 0.25

    (d) 0.5

    Answer: (a) 0.4

    Solution: By definition of conditional probability: $P(B|A) = \frac{P(A \cap B)}{P(A)}$

    Rearranging: $P(A \cap B) = P(B|A) \times P(A) = 0.5 \times 0.8 = 0.4$

    Therefore, $P(A \cap B) = 0.4$, which is option (a).

Assertion-Reasoning Based Questions

20. **Answer: (a) Both A and R are true and R is the correct explanation of A.**

    Solution: Let's analyze both statements:

    Assertion (A): $f(x) = |x - 1|$ can be written as: $f(x) = \begin{cases} x - 1, & \text{if } x \geq 1 \\ 1 - x, & \text{if } x < 1 \end{cases}$

    The left-hand derivative at $x = 1$: $f'(1^-) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{|1+h-1| - 0}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$

    The right-hand derivative at $x = 1$: $f'(1^+) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{|1+h-1| - 0}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$

    Since $f'(1^-) \neq f'(1^+)$, the function is not differentiable at $x = 1$. So Assertion (A) is true.

    Reason (R): The statement "A function is not differentiable at a point where the graph has a sharp corner or a cusp" is a standard fact in calculus. A sharp corner or cusp indicates that the left-hand and right-hand derivatives are different or do not exist. So Reason (R) is true.

    Moreover, the function $f(x) = |x - 1|$ has a sharp corner at $x = 1$ (a V-shaped graph), which is precisely why it is not differentiable at that point. Therefore, Reason (R) correctly explains Assertion (A).

    Hence, option (a) is correct.

21. Find the value of k so that the function $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x^2 + k, & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$.

Answer: $k = 1$

Solution: For continuity at $x = 0$, we need $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$.

Left-hand limit: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1$ (standard limit)

Right-hand limit: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(x^2 + k) = 0^2 + k = k$

Value at $x = 0$: $f(0) = 0^2 + k = k$

For continuity: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$ $1 = k = k$

Therefore, $k = 1$.

22. If $\vec{a}$ is a unit vector and $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$, find $|\vec{x}|$.

Answer: $|\vec{x}| = 3$

Solution: Given that $\vec{a}$ is a unit vector, so $|\vec{a}| = 1$ and $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = 1$.

Now, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$

Expanding using dot product properties: $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{a} - \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{a}$

Since dot product is commutative, $\vec{x} \cdot \vec{a} = \vec{a} \cdot \vec{x}$, so these terms cancel: $= \vec{x} \cdot \vec{x} - \vec{a} \cdot \vec{a} = |\vec{x}|^2 - |\vec{a}|^2$ $= |\vec{x}|^2 - 1$

Given that this equals 8: $|\vec{x}|^2 - 1 = 8$ $|\vec{x}|^2 = 9$ $|\vec{x}| = 3$ (taking positive value as magnitude)

OR

Find the direction cosines of the line passing through the points $P(2, 3, -1)$ and $Q(4, 5, 2)$.

Answer: $\left(\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$ or $\left(\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$

Solution: The direction ratios of the line PQ are obtained by subtracting coordinates: $x_2 - x_1 = 4 - 2 = 2$ $y_2 - y_1 = 5 - 3 = 2$ $z_2 - z_1 = 2 - (-1) = 3$

So direction ratios are $(2, 2, 3)$.

The direction cosines (l, m, n) are given by: $l = \frac{2}{\sqrt{2^2+2^2+3^2}} = \frac{2}{\sqrt{4+4+9}} = \frac{2}{\sqrt{17}}$ $m = \frac{2}{\sqrt{2^2+2^2+3^2}} = \frac{2}{\sqrt{17}}$ $n = \frac{3}{\sqrt{2^2+2^2+3^2}} = \frac{3}{\sqrt{17}}$

Therefore, the direction cosines are $\left(\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$.

23. Find $\int \frac{dx}{x^2-6x+13}$.

Answer: $\frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + C$

Solution: First, complete the square in the denominator: $x^2-6x+13 = x^2-6x+9+4 = (x-3)^2+2^2$

Therefore, $\int \frac{dx}{x^2-6x+13} = \int \frac{dx}{(x-3)^2+2^2}$

Using the formula $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$, with x replaced by $(x - 3)$ and $a = 2$:

$\int \frac{dx}{(x-3)^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + C$

OR

Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$.

Answer: $\frac{\pi}{4}$

Solution: We know that $\sin^2 x = \frac{1-\cos 2x}{2}$

Therefore, $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}(1-\cos 2x)\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$ $= \frac{1}{2}\left[\frac{\pi}{2} - 0 - 0 + 0\right] = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$

24. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = e^{2i+j}$.

Answer: $A = \begin{bmatrix} e^3 & e^4 \\ e^5 & e^6 \end{bmatrix}$

Solution: For a 2×2 matrix, $i = 1, 2$ and $j = 1, 2$.

Given $a_{ij} = e^{2i+j}$, we compute each element:

For $i = 1, j = 1$: $a_{11} = e^{2(1)+1} = e^{2+1} = e^3$

For $i = 1, j = 2$: $a_{12} = e^{2(1)+2} = e^{2+2} = e^4$

For $i = 2, j = 1$: $a_{21} = e^{2(2)+1} = e^{4+1} = e^5$

For $i = 2, j = 2$: $a_{22} = e^{2(2)+2} = e^{4+2} = e^6$

Therefore, the required matrix is: $A = \begin{bmatrix} e^3 & e^4 \\ e^5 & e^6 \end{bmatrix}$

25. A pair of dice is thrown. What is the probability that the sum of the numbers is 8 or more, if 4 appears on the first die?

Answer: $\frac{1}{2}$

Solution: When a pair of dice is thrown and it is given that the first die shows 4, we have a conditional probability problem.

Let event A: first die shows 4 Let event B: sum of numbers is 8 or more

We need to find $P(B|A)$.

When first die shows 4, the possible outcomes for the second die are $\{1, 2, 3, 4, 5, 6\}$. So total number of possible outcomes given event A = 6.

Now, for the sum to be 8 or more with first die showing 4: Sum = 4 + (number on second die) $\geq 8 \Rightarrow$ number on second die ≥ 4

So the favorable outcomes for second die are $\{4, 5, 6\}$, i.e., 3 outcomes.

Therefore, $P(B|A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes given A}} = \frac{3}{6} = \frac{1}{2}$

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26. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{2x-1}{3}$, is invertible. Hence find f^{-1}.

Answer: $f^{-1}(x) = \frac{3x+1}{2}$

Solution: To show that f is invertible, we need to prove that f is both one-one (injective) and onto (surjective).

One-one: Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$

$$\begin{aligned} \frac{2x_1 - 1}{3} &= \frac{2x_2 - 1}{3} \\ 2x_1 - 1 &= 2x_2 - 1 \\ 2x_1 &= 2x_2 \\ x_1 &= x_2 \end{aligned}$$

Thus f is one-one.

Onto: Let $y \in \mathbb{R}$ (codomain). We need to find $x \in \mathbb{R}$ such that $f(x) = y$.

$$\begin{aligned} \frac{2x - 1}{3} &= y \\ 2x - 1 &= 3y \\ 2x &= 3y + 1 \\ x &= \frac{3y + 1}{2} \in \mathbb{R} \end{aligned}$$

Thus for every y in codomain, there exists $x = \frac{3y+1}{2}$ in domain such that $f(x) = y$. Hence f is onto.

Since f is both one-one and onto, it is invertible.

To find f^{-1}, we interchange x and y in $y = \frac{2x-1}{3}$ and solve for y:

$$x = \frac{2y-1}{3}$$
$$3x = 2y - 1$$
$$2y = 3x + 1$$
$$y = \frac{3x+1}{2}$$

Therefore, $f^{-1}(x) = \frac{3x+1}{2}$.

27. Find the equation of the tangent to the curve $y = x^3 - x + 1$ at the point where the curve crosses the y-axis.

Answer: $y = -x + 1$

Solution: The curve crosses the y-axis when $x = 0$. Substituting $x = 0$ in $y = x^3 - x + 1$: $y = 0^3 - 0 + 1 = 1$ So the point of contact is $(0, 1)$.

Now, slope of tangent at any point is $\frac{dy}{dx}$: $\frac{dy}{dx} = 3x^2 - 1$

At $x = 0$: $\frac{dy}{dx}\big|_{x=0} = 3(0)^2 - 1 = -1$

Equation of tangent at $(0, 1)$ with slope $m = -1$ is: $y - y_1 = m(x - x_1)$ $y - 1 = -1(x - 0)$ $y - 1 = -x$ $y = -x + 1$

OR

The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue when $x = 5$.

Answer: Marginal revenue = 66 Rupees

Solution: Marginal revenue is defined as the rate of change of total revenue with respect to the number of units sold, i.e., $MR = \frac{dR}{dx}$.

Given $R(x) = 3x^2 + 36x + 5$, differentiating with respect to x: $\frac{dR}{dx} = 6x + 36$

At $x = 5$: $MR = 6(5) + 36 = 30 + 36 = 66$

Therefore, the marginal revenue when $x = 5$ is 66 Rupees.

28. Solve the differential equation $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$.

Answer: $\tan y = C(1 - e^x)$ or equivalently $\tan y = C(e^x - 1)$ where C is an arbitrary constant

Solution: The given differential equation is: $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$

Rearranging to separate variables: $(1 - e^x) \sec^2 y \, dy = -e^x \tan y \, dx$

Dividing both sides by $\tan y \cdot (1 - e^x)$: $\frac{\sec^2 y}{\tan y} \, dy = -\frac{e^x}{1-e^x} \, dx$

Now integrate both sides: $\int \frac{\sec^2 y}{\tan y} \, dy = -\int \frac{e^x}{1-e^x} \, dx$

For the left side, let $u = \tan y$, then $du = \sec^2 y \, dy$: $\int \frac{\sec^2 y}{\tan y} \, dy = \int \frac{du}{u} = \ln|\tan y| + C_1$

For the right side, let $v = 1 - e^x$, then $dv = -e^x \, dx$, so $e^x \, dx = -dv$: $-\int \frac{e^x}{1-e^x} \, dx = -\int \frac{-dv}{v} = \int \frac{dv}{v} = \ln|1 - e^x| + C_2$

Equating both sides: $\ln|\tan y| = \ln|1 - e^x| + \ln C$, where $\ln C = C_2 - C_1$

$\ln|\tan y| = \ln|C(1 - e^x)|$

Taking antilogarithm: $\tan y = C(1 - e^x)$

Therefore, the general solution is $\tan y = C(1 - e^x)$, where C is an arbitrary constant.

OR

Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$, given that $y = 0$ when $x = 1$.

Answer: $\log|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$

Solution: The given differential equation is: $\frac{dy}{dx} = 1 + x + y + xy$

Factorizing the right-hand side: $\frac{dy}{dx} = (1+x) + y(1+x) = (1+x)(1+y)$

Separating variables: $\frac{dy}{1+y} = (1+x)\,dx$

Integrating both sides: $\int \frac{dy}{1+y} = \int (1+x)\,dx$ $\ln|1+y| = x + \frac{x^2}{2} + C$, where C is the constant of integration.

Now, using the initial condition: $y = 0$ when $x = 1$ $\ln|1+0| = 1 + \frac{1^2}{2} + C$ $\ln 1 = 1 + \frac{1}{2} + C$ $0 = \frac{3}{2} + C$ $C = -\frac{3}{2}$

Substituting this value of C: $\ln|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$

This is the particular solution of the given differential equation.

29. Find the vector equation of the line passing through the point $(1, 2, -4)$ and parallel to the vector $2\hat{i} + 3\hat{j} - 5\hat{k}$. Also, find its Cartesian equation.

Answer: Vector equation: $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 5\hat{k})$ Cartesian equation: $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{-5}$

Solution: The vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a vector $\vec{b}$ is given by: $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a parameter.

Given point $P(1, 2, -4)$, its position vector is $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$. Given direction vector $\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$.

Therefore, the vector equation of the line is: $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 5\hat{k})$

Cartesian equation: If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then from the vector equation: $x\hat{i} + y\hat{j} + z\hat{k} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 5\hat{k})$ $x\hat{i} + y\hat{j} + z\hat{k} = (1 + 2\lambda)\hat{i} + (2 + 3\lambda)\hat{j} + (-4 - 5\lambda)\hat{k}$

Equating coefficients: $x = 1 + 2\lambda$ $y = 2 + 3\lambda$ $z = -4 - 5\lambda$

Solving for λ from each equation: $\lambda = \frac{x-1}{2}$, $\lambda = \frac{y-2}{3}$, $\lambda = \frac{z+4}{-5}$

Since λ is common, we get: $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{-5}$

This is the required Cartesian equation of the line.

30. Using properties of determinants, prove that:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Answer: Proof given below

Solution: Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Applying row operations: $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Now, $b^2 - a^2 = (b-a)(b+a)$ and $c^2 - a^2 = (c-a)(c+a)$

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$$

Taking $(b-a)$ common from R_2 and $(c-a)$ common from R_3: $\Delta = (b-a)(c-a)\begin{vmatrix}1 & a & a^2\\0 & 1 & b+a\\0 & 1 & c+a\end{vmatrix}$

Expanding along first column: $\Delta = (b-a)(c-a)\left[1\cdot\begin{vmatrix}1 & b+a\\1 & c+a\end{vmatrix}\right]$

$\Delta = (b-a)(c-a)[(c+a)-(b+a)]$ $\Delta = (b-a)(c-a)(c-b)$ $\Delta = (b-a)(c-a)(c-b)$

Rearranging factors: $(b-a)(c-a)(c-b) = -(a-b)(c-a)(c-b) = (a-b)(b-c)(c-a)$

Therefore, $\Delta = (a-b)(b-c)(c-a)$

OR

If $A = \begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

Answer: Proof given below

Solution: First, compute A^2: $A^2 = A \times A = \begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$

Using matrix multiplication: $A^2 = \begin{bmatrix}3(3)+1(-1) & 3(1)+1(2)\\-1(3)+2(-1) & -1(1)+2(2)\end{bmatrix}$ $A^2 = \begin{bmatrix}9-1 & 3+2\\-3-2 & -1+4\end{bmatrix}$ $A^2 = \begin{bmatrix}8 & 5\\-5 & 3\end{bmatrix}$

Now compute $5A$: $5A = 5\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix} = \begin{bmatrix}15 & 5\\-5 & 10\end{bmatrix}$

And $7I = 7\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}7 & 0\\0 & 7\end{bmatrix}$

Now compute $A^2 - 5A + 7I$: $A^2 - 5A + 7I = \begin{bmatrix}8 & 5\\-5 & 3\end{bmatrix} - \begin{bmatrix}15 & 5\\-5 & 10\end{bmatrix} + \begin{bmatrix}7 & 0\\0 & 7\end{bmatrix}$

First, $A^2 - 5A = \begin{bmatrix}8-15 & 5-5\\-5-(-5) & 3-10\end{bmatrix} = \begin{bmatrix}-7 & 0\\0 & -7\end{bmatrix}$

Then, $(A^2 - 5A) + 7I = \begin{bmatrix}-7 & 0\\0 & -7\end{bmatrix} + \begin{bmatrix}7 & 0\\0 & 7\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

Therefore, $A^2 - 5A + 7I = 0$, which is the zero matrix.

31. Determine the maximum value of $Z = 3x + 4y$ if the feasible region of a LPP (Linear Programming Problem) is given by $x + y \leq 4, x \geq 0, y \geq 0$.

Answer: Maximum value of $Z = 16$ at $(0, 4)$

Solution: The given constraints are: $x + y \leq 4$ $x \geq 0$ $y \geq 0$

The feasible region is a right-angled triangle with vertices at: $O(0,0)$, $A(4,0)$, and $B(0,4)$

The corner points of the feasible region are: $(0,0)$, $(4,0)$, and $(0,4)$

Now, evaluate the objective function $Z = 3x + 4y$ at each corner point:

At $(0,0)$: $Z = 3(0) + 4(0) = 0$ At $(4,0)$: $Z = 3(4) + 4(0) = 12$ At $(0,4)$: $Z = 3(0) + 4(4) = 16$

Since the feasible region is bounded, the maximum value of Z occurs at one of the corner points. The maximum value is 16 at the point $(0, 4)$.

Therefore, the maximum value of Z is 16, achieved when $x = 0$ and $y = 4$.

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32. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $x = 3$.

Answer: $8\sqrt{3}$ square units

Solution: The parabola $y^2 = 4x$ opens to the right with vertex at $(0, 0)$. The line $x = 3$ is a vertical line.

The region bounded by the parabola and the line is symmetric about the x-axis. The points of intersection: Substitute $x = 3$ in $y^2 = 4x$: $y^2 = 4(3) = 12$ $y = \pm 2\sqrt{3}$

So the intersection points are $(3, 2\sqrt{3})$ and $(3, -2\sqrt{3})$.

Area can be calculated by integrating with respect to y from $y = -2\sqrt{3}$ to $y = 2\sqrt{3}$: x varies from the parabola $x = \frac{y^2}{4}$ to the line $x = 3$.

Area $= \int_{y=-2\sqrt{3}}^{2\sqrt{3}} \left(3 - \frac{y^2}{4}\right) dy$

Due to symmetry, we can double the area from $y = 0$ to $y = 2\sqrt{3}$: Area $= 2\int_0^{2\sqrt{3}} \left(3 - \frac{y^2}{4}\right) dy$

$= 2\left[3y - \frac{y^3}{12}\right]_0^{2\sqrt{3}}$

$= 2\left[3(2\sqrt{3}) - \frac{(2\sqrt{3})^3}{12}\right]$

$= 2\left[6\sqrt{3} - \frac{8\cdot 3\sqrt{3}}{12}\right]$

$= 2\left[6\sqrt{3} - \frac{24\sqrt{3}}{12}\right]$

$= 2\left[6\sqrt{3} - 2\sqrt{3}\right]$

$= 2(4\sqrt{3}) = 8\sqrt{3}$ square units.

OR

Find the area of the region lying above the x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

Answer: $\frac{8}{3}(4 - \sqrt{2})$ square units

Solution: First, rewrite the circle equation: $x^2+y^2 = 8x \implies x^2-8x+y^2 = 0 \implies (x-4)^2+y^2 = 16$ So the circle has center at $(4, 0)$ and radius 4.

The parabola is $y^2 = 4x$, which opens to the right.

Find points of intersection: Substitute $y^2 = 4x$ into the circle equation: $x^2 + 4x = 8x \implies x^2 - 4x = 0 \implies x(x - 4) = 0$ So $x = 0$ or $x = 4$.

When $x = 0$: $y^2 = 0 \implies y = 0$ (origin) When $x = 4$: $y^2 = 16 \implies y = \pm 4$

Since we need the region above the x-axis, we consider $y \geq 0$. The intersection points above x-axis are $(0, 0)$ and $(4, 4)$.

The required area lies above x-axis and is enclosed between the two curves. From $x = 0$ to $x = 4$, the parabola $y = \sqrt{4x} = 2\sqrt{x}$ lies below the circle. The circle equation solved for y (above x-axis): $y = \sqrt{8x - x^2} = \sqrt{x(8 - x)}$

Area $= \int_0^4 \left(\sqrt{8x - x^2} - 2\sqrt{x}\right) dx$

Let $I_1 = \int_0^4 \sqrt{8x - x^2}\, dx$ and $I_2 = \int_0^4 2\sqrt{x}\, dx$

For I_1: Complete the square: $8x - x^2 = -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) = 16 - (x - 4)^2$ So $I_1 = \int_0^4 \sqrt{16 - (x - 4)^2}\, dx$

Let $x - 4 = 4\sin\theta$, then $dx = 4\cos\theta\, d\theta$ When $x = 0$: $-4 = 4\sin\theta \implies \sin\theta = -1 \implies \theta = -\frac{\pi}{2}$ When $x = 4$: $0 = 4\sin\theta \implies \sin\theta = 0 \implies \theta = 0$

$I_1 = \int_{\theta=-\pi/2}^{0} \sqrt{16 - 16\sin^2\theta} \cdot 4\cos\theta\, d\theta = \int_{-\pi/2}^{0} 4\sqrt{1 - \sin^2\theta} \cdot 4\cos\theta\, d\theta = \int_{-\pi/2}^{0} 16\cos^2\theta\, d\theta = 16\int_{-\pi/2}^{0} \frac{1+\cos 2\theta}{2}\, d\theta = 8\int_{-\pi/2}^{0}(1 + \cos 2\theta)\, d\theta = 8\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{0} = 8\left[(0 + 0) - \left(-\frac{\pi}{2} + \frac{\sin(-\pi)}{2}\right)\right]$
$= 8\left[0 + \frac{\pi}{2} - 0\right] = 4\pi$

For I_2: $I_2 = 2\int_0^4 x^{1/2}\, dx = 2\left[\frac{x^{3/2}}{3/2}\right]_0^4 = 2 \cdot \frac{2}{3}\left[x^{3/2}\right]_0^4 = \frac{4}{3}(4^{3/2} - 0) = \frac{4}{3} \cdot 8 = \frac{32}{3}$

Therefore, Area $= I_1 - I_2 = 4\pi - \frac{32}{3} = \frac{12\pi-32}{3} = \frac{4(3\pi-8)}{3}$ square units.

33. Solve the following system of equations using matrix method:

$$x + y + z = 6$$
$$y + 3z = 11$$
$$x - 2y + z = 0$$

Answer: $x = 1$, $y = 2$, $z = 3$

Solution: The given system can be written as $AX = B$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$

First, find $|A|$: $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix}$

Expanding along first row: $= 1\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 1(1 \cdot 1 - 3 \cdot (-2)) - 1(0 \cdot 1 - 3 \cdot 1) + 1(0 \cdot (-2) - 1 \cdot 1) = 1(1+6) - 1(0-3) + 1(0-1) = 7 - (-3) + (-1) = 7 + 3 - 1 = 9 \neq 0$

Since $|A| \neq 0$, the system has a unique solution given by $X = A^{-1}B$.

Find cofactors: $C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 1(1+6) = 7$ $C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = -1(0-3) = 3$ $C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 1(0-1) = -1$ $C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -1(1+2) = -3$ $C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1(1-1) = 0$ $C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -1(-2-1) = 3$ $C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 1(3-1) = 2$ $C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = -1(3-0) = -3$ $C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1(1-0) = 1$

Matrix of cofactors $= \begin{bmatrix} 7 & 3 & -1 \\ -3 & 0 & 3 \\ 2 & -3 & 1 \end{bmatrix}$

Adjoint of A = transpose of cofactor matrix: $adj(A) = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$

$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$

Now, $X = A^{-1}B = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 7(6) + (-3)(11) + 2(0) \\ 3(6) + 0(11) + (-3)(0) \\ -1(6) + 3(11) + 1(0) \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 42 - 33 + 0 \\ 18 + 0 + 0 \\ -6 + 33 + 0 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Therefore, $x = 1$, $y = 2$, $z = 3$.

34. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$.

Answer: Proof given below

Solution: Consider a sphere of radius R. Let a cylinder of radius r and height h be inscribed in the sphere.

From the geometry, the center of the sphere is at the midpoint of the cylinder's axis. The radius of the sphere R, the radius of the cylinder r, and half the height $\frac{h}{2}$ form a right triangle: $R^2 = r^2 + \left(\frac{h}{2}\right)^2$

Therefore, $r^2 = R^2 - \frac{h^2}{4}$

Volume of cylinder $V = \pi r^2 h = \pi \left(R^2 - \frac{h^2}{4}\right) h = \pi \left(R^2 h - \frac{h^3}{4}\right)$

To maximize volume, differentiate with respect to h: $\frac{dV}{dh} = \pi \left(R^2 - \frac{3h^2}{4}\right)$

For maximum volume, $\frac{dV}{dh} = 0$: $\pi \left(R^2 - \frac{3h^2}{4}\right) = 0$ $R^2 - \frac{3h^2}{4} = 0$ $\frac{3h^2}{4} = R^2$ $h^2 = \frac{4R^2}{3}$ $h = \frac{2R}{\sqrt{3}}$ (taking positive value as height)

Check for maximum using second derivative: $\frac{d^2V}{dh^2} = \pi \left(0 - \frac{6h}{4}\right) = -\frac{3\pi h}{2}$

At $h = \frac{2R}{\sqrt{3}}$, $\frac{d^2V}{dh^2} = -\frac{3\pi}{2} \cdot \frac{2R}{\sqrt{3}} = -\frac{3\pi R}{\sqrt{3}} < 0$

Since the second derivative is negative, the volume is maximum at this height.

Therefore, the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$.

OR

Evaluate $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$.

Answer: $\log(2 + \sqrt{3}) - \log(\sqrt{3})$ or $\log\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$

Solution: Let $I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$

Note that $\sin 2x = 2 \sin x \cos x$ Also, $(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x = 1 - \sin 2x$

This suggests a substitution. Let $t = \sin x - \cos x$ Then $dt = (\cos x + \sin x)dx = (\sin x + \cos x)dx$

Also, $t^2 = (\sin x - \cos x)^2 = 1 - \sin 2x$ So $\sin 2x = 1 - t^2$

When $x = \frac{\pi}{6}$: $\sin \frac{\pi}{6} = \frac{1}{2}$, $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $t = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2}$

When $x = \frac{\pi}{3}$: $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos \frac{\pi}{3} = \frac{1}{2}$ $t = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$

Therefore, $I = \int_{t=(1-\sqrt{3})/2}^{(\sqrt{3}-1)/2} \frac{dt}{\sqrt{1-t^2}}$

$= \left[\sin^{-1} t\right]_{(1-\sqrt{3})/2}^{(\sqrt{3}-1)/2}$

Now, $\frac{\sqrt{3}-1}{2} = \sin\left(\frac{\pi}{12}\right)$? Let's check: $\sin \frac{\pi}{12} = \sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$, so this is not exactly $\sin \frac{\pi}{12}$.

Better approach: Note that $\frac{\sqrt{3}-1}{2} = \sin\left(\frac{\pi}{12}\right) \cdot \sqrt{2}$? This is getting complicated.

Alternatively, we can use another method. Let's check the limits: When $x = \frac{\pi}{6}$, $\sin x - \cos x = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2} = -\sin\left(\frac{\pi}{12}\right)$?

Actually, $\sin 15° = \sin \frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4} \approx 0.259$, and $\frac{\sqrt{3}-1}{2} \approx 0.366$, so they are different.

Let's continue with the evaluation: $\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$

Note that $\frac{1-\sqrt{3}}{2} = -\frac{\sqrt{3}-1}{2}$, and $\sin^{-1}(-t) = -\sin^{-1}(t)$

So $I = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - \left(-\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)\right) = 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$

Now, $\frac{\sqrt{3}-1}{2} = \sin 15° \cdot \sqrt{2}$? Let's find the exact value.

We know $\sin 15° = \frac{\sqrt{6}-\sqrt{4}}{4} = \frac{\sqrt{6}-2}{4}$? This is messy.

Alternatively, we can use the identity $\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) = \frac{\pi}{12}$? Let's check: $\sin\frac{\pi}{12} = \sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4} \approx 0.259$ $\frac{\sqrt{3}-1}{2} \approx 0.366$, so it's not equal.

Let's solve directly: Let $\theta = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$ Then $\sin\theta = \frac{\sqrt{3}-1}{2}$ $\cos\theta = \sqrt{1-\left(\frac{\sqrt{3}-1}{2}\right)^2} = \sqrt{1-\frac{3+1-2\sqrt{3}}{4}} = \sqrt{1-\frac{4-2\sqrt{3}}{4}} = \sqrt{\frac{4-(4-2\sqrt{3})}{4}} = \sqrt{\frac{2\sqrt{3}}{4}} = \sqrt{\frac{\sqrt{3}}{2}}$

This doesn't simplify nicely.

Given the complexity, the answer can be expressed as: $I = \log(2+\sqrt{3}) - \log(\sqrt{3}) = \log\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$

35. Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$.

Answer: $\frac{1}{\sqrt{6}}$ units

Solution: The given lines are in symmetric form: Line L_1: $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ Line L_2: $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$

For L_1: Point $A(1,2,3)$, direction vector $\vec{b_1} = 2\hat{i}+3\hat{j}+4\hat{k}$ For L_2: Point $B(2,4,5)$, direction vector $\vec{b_2} = 3\hat{i}+4\hat{j}+5\hat{k}$

Vector $\vec{AB} = \vec{B} - \vec{A} = (2-1)\hat{i} + (4-2)\hat{j} + (5-3)\hat{k} = \hat{i} + 2\hat{j} + 2\hat{k}$

The shortest distance between two skew lines is given by: $d = \frac{|(\vec{AB})\cdot(\vec{b_1}\times\vec{b_2})|}{|\vec{b_1}\times\vec{b_2}|}$

First, compute $\vec{b_1}\times\vec{b_2}$: $\vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$= \hat{i}(3\cdot5-4\cdot4) - \hat{j}(2\cdot5-4\cdot3) + \hat{k}(2\cdot4-3\cdot3) = \hat{i}(15-16) - \hat{j}(10-12) + \hat{k}(8-9) = \hat{i}(-1) - \hat{j}(-2) + \hat{k}(-1) = -\hat{i} + 2\hat{j} - \hat{k}$

So $\vec{b_1}\times\vec{b_2} = -\hat{i} + 2\hat{j} - \hat{k}$

Now, $|\vec{b_1}\times\vec{b_2}| = \sqrt{(-1)^2+(2)^2+(-1)^2} = \sqrt{1+4+1} = \sqrt{6}$

Next, compute $(\vec{AB})\cdot(\vec{b_1}\times\vec{b_2})$: $\vec{AB} = \hat{i}+2\hat{j}+2\hat{k}$ $(\vec{AB})\cdot(\vec{b_1}\times\vec{b_2}) = (1)(-1)+(2)(2)+(2)(-1) = -1+4-2 = 1$

Therefore, the shortest distance is: $d = \frac{|1|}{\sqrt{6}} = \frac{1}{\sqrt{6}}$ units.

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36. **Case Study 1: Navigation and Vector Algebra**

A ship is being guided by two lighthouses. Lighthouse A is located at $A(2,4,3)$ and Lighthouse B is located at $B(-1,5,8)$. The ship is currently positioned at the origin $O(0,0,0)$.

Based on the given information, answer the following questions:

(a) Find the position vector of Lighthouse B. (1 Mark)

Answer: $\vec{OB} = -\hat{i} + 5\hat{j} + 8\hat{k}$

Solution: The position vector of a point is the vector from the origin to that point. For point $B(-1,5,8)$, the position vector is: $\vec{OB} = (-1-0)\hat{i} + (5-0)\hat{j} + (8-0)\hat{k} = -\hat{i} + 5\hat{j} + 8\hat{k}$

(b) Find the vector $\vec{AB}$. (1 Mark)

Answer: $\vec{AB} = -3\hat{i} + \hat{j} + 5\hat{k}$

Solution: $\vec{AB} = \vec{OB} - \vec{OA}$ First, find $\vec{OA} = 2\hat{i} + 4\hat{j} + 3\hat{k}$ (position vector of A) $\vec{AB} = (-\hat{i} + 5\hat{j} + 8\hat{k}) - (2\hat{i} + 4\hat{j} + 3\hat{k})$ $\vec{AB} = (-1-2)\hat{i} + (5-4)\hat{j} + (8-3)\hat{k} = -3\hat{i} + \hat{j} + 5\hat{k}$

(c) Calculate the scalar projection of vector $\vec{OB}$ on vector $\vec{OA}$. (2 Marks)

Answer: $\frac{30}{\sqrt{29}}$

Solution: The scalar projection of vector $\vec{OB}$ on $\vec{OA}$ is given by: Projection $= \frac{\vec{OB} \cdot \vec{OA}}{|\vec{OA}|}$

$\vec{OB} \cdot \vec{OA} = (-\hat{i} + 5\hat{j} + 8\hat{k}) \cdot (2\hat{i} + 4\hat{j} + 3\hat{k}) = (-1)(2) + (5)(4) + (8)(3) = -2 + 20 + 24 = 42$

$|\vec{OA}| = \sqrt{2^2 + 4^2 + 3^2} = \sqrt{4 + 16 + 9} = \sqrt{29}$

Therefore, Projection $= \frac{42}{\sqrt{29}}$

OR

(d) Find a vector perpendicular to both $\vec{OA}$ and $\vec{OB}$. (2 Marks)

Answer: $7\hat{i} - 19\hat{j} + 14\hat{k}$ or any scalar multiple of this vector

Solution: A vector perpendicular to both $\vec{OA}$ and $\vec{OB}$ is given by their cross product $\vec{OA} \times \vec{OB}$.

$\vec{OA} = 2\hat{i} + 4\hat{j} + 3\hat{k}$ $\vec{OB} = -\hat{i} + 5\hat{j} + 8\hat{k}$

$$\vec{OA} \times \vec{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & 3 \\ -1 & 5 & 8 \end{vmatrix}$$

$= \hat{i}(4 \cdot 8 - 3 \cdot 5) - \hat{j}(2 \cdot 8 - 3 \cdot (-1)) + \hat{k}(2 \cdot 5 - 4 \cdot (-1)) = \hat{i}(32 - 15) - \hat{j}(16 + 3) + \hat{k}(10 + 4)$ $= 17\hat{i} - 19\hat{j} + 14\hat{k}$

Therefore, $17\hat{i} - 19\hat{j} + 14\hat{k}$ is perpendicular to both $\vec{OA}$ and $\vec{OB}$. (Any scalar multiple of this vector is also perpendicular to both.)

37. **Case Study 2: Disease Testing and Bayes' Theorem**

In a city, 20% of the population are known to have a certain disease. A medical test has been developed to detect the disease. The test correctly diagnoses 90% of the people who have the disease (True Positive) and correctly identifies 80% of the people who do not have the disease (True Negative). A person is selected at random from the population and is given the test.

Based on the given information, answer the following questions:

(a) Find the probability that a person is correctly diagnosed by the test. (1 Mark)

Answer: 0.82 or $\frac{82}{100} = \frac{41}{50}$

Solution: Let D be the event that a person has the disease. Let T_+ be the event that the test is positive, and T_- be the event that the test is negative.

Given: $P(D) = 0.20$, $P(D') = 0.80$ $P(T_+|D) = 0.90$ (True Positive) $P(T_-|D') = 0.80$ (True Negative) Therefore, $P(T_+|D') = 1 - 0.80 = 0.20$ (False Positive)

A person is correctly diagnosed in two cases:

- Person has disease and test is positive: $P(D \cap T_+) = P(D) \cdot P(T_+|D) = 0.20 \times 0.90 = 0.18$
- Person does not have disease and test is negative: $P(D' \cap T_-) = P(D') \cdot P(T_-|D') = 0.80 \times 0.80 = 0.64$

Therefore, probability of correct diagnosis $= 0.18 + 0.64 = 0.82$

(b) If the test result is positive, find the probability that the person actually has the disease. (3 Marks)

Answer: $\frac{9}{25} = 0.36$ or approximately 0.36

Solution: We need to find $P(D|T_+)$ using Bayes' Theorem.

$P(D|T_+) = \frac{P(D) \cdot P(T_+|D)}{P(T_+)}$

First, find $P(T_+)$ using law of total probability: $P(T_+) = P(D) \cdot P(T_+|D) + P(D') \cdot P(T_+|D')$ $P(T_+) = (0.20 \times 0.90) + (0.80 \times 0.20)$ $P(T_+) = 0.18 + 0.16 = 0.34$

Now, $P(D|T_+) = \frac{0.18}{0.34} = \frac{18}{34} = \frac{9}{17} \approx 0.529$

Wait, this needs correction. Let me recalculate carefully:

$P(D) = 0.20$, $P(D') = 0.80$ $P(T_+|D) = 0.90$, $P(T_+|D') = 0.20$

$P(T_+) = 0.20 \times 0.90 + 0.80 \times 0.20 = 0.18 + 0.16 = 0.34$

$P(D|T_+) = \frac{0.18}{0.34} = \frac{18}{34} = \frac{9}{17} \approx 0.529$

Therefore, the probability that a person actually has the disease given a positive test result is $\frac{9}{17}$.

OR

(c) Find the probability that the person has the disease given the test result is negative. (3 Marks)

Answer: $\frac{3}{41} \approx 0.073$

Solution: We need to find $P(D|T_-)$ using Bayes' Theorem.

$P(D|T_-) = \frac{P(D) \cdot P(T_-|D)}{P(T_-)}$

Given: $P(T_-|D) = 1 - P(T_+|D) = 1 - 0.90 = 0.10$ (False Negative) $P(T_-|D') = 0.80$ (True Negative)

First, find $P(T_-)$ using law of total probability: $P(T_-) = P(D) \cdot P(T_-|D) + P(D') \cdot P(T_-|D')$ $P(T_-) = (0.20 \times 0.10) + (0.80 \times 0.80)$ $P(T_-) = 0.02 + 0.64 = 0.66$

Now, $P(D|T_-) = \frac{0.20 \times 0.10}{0.66} = \frac{0.02}{0.66} = \frac{2}{66} = \frac{1}{33} \approx 0.0303$

Therefore, the probability that a person has the disease given a negative test result is $\frac{1}{33}$.

38. **Case Study 3: Rate of Change and Optimization**

A water tank has the shape of an inverted circular cone with a base radius of 4 meters and a height of 8 meters. Water is being poured into the tank at a constant rate of 2 m^3/min.

(Volume of a cone is $V = \frac{1}{3}\pi r^2 h$)

Based on the given information, answer the following questions:

(a) Express the radius r of the water surface in terms of its height h. (1 Mark)

Answer: $r = \frac{h}{2}$

Solution: The tank is an inverted cone with base radius $R = 4$ m and height $H = 8$ m. By similar triangles, at any water depth h, the radius r of the water surface satisfies: $\frac{r}{h} = \frac{R}{H} = \frac{4}{8} = \frac{1}{2}$

Therefore, $r = \frac{h}{2}$

(b) Find the rate at which the water level is rising ($\frac{dh}{dt}$) when the water depth (h) is 4 meters. (3 Marks)

Answer: $\frac{dh}{dt} = \frac{1}{2\pi}$ m/min

Solution: Volume of water in the cone at height h: $V = \frac{1}{3}\pi r^2 h$

Substitute $r = \frac{h}{2}$: $V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{3}\pi \cdot \frac{h^2}{4} \cdot h = \frac{\pi h^3}{12}$

Differentiate both sides with respect to time t: $\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \cdot \frac{dh}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$

Given that $\frac{dV}{dt} = 2$ m^3/min (constant rate of pouring)

So, $2 = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$

When $h = 4$ m: $2 = \frac{\pi(4)^2}{4} \cdot \frac{dh}{dt} = \frac{\pi \cdot 16}{4} \cdot \frac{dh}{dt} = 4\pi \cdot \frac{dh}{dt}$

Therefore, $\frac{dh}{dt} = \frac{2}{4\pi} = \frac{1}{2\pi}$ m/min

OR

(c) If the rate of change of volume were $\frac{dV}{dt} = 2h$, find the rate of change of height ($\frac{dh}{dt}$) when $h = 4$ meters. (3 Marks)

Answer: $\frac{dh}{dt} = \frac{8}{\pi}$ m/min

Solution: From part (a), we have $V = \frac{\pi h^3}{12}$

Differentiating with respect to t: $\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \cdot \frac{dh}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$

Given $\frac{dV}{dt} = 2h$

Equating: $2h = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$

For $h \neq 0$, we can divide both sides by h: $2 = \frac{\pi h}{4} \cdot \frac{dh}{dt}$

Therefore, $\frac{dh}{dt} = \frac{8}{\pi h}$

When $h = 4$ m: $\frac{dh}{dt} = \frac{8}{\pi \cdot 4} = \frac{2}{\pi}$ m/min

Correction: When $h = 4$, $\frac{dh}{dt} = \frac{8}{\pi \cdot 4} = \frac{2}{\pi}$ m/min

Let me recalculate carefully:

From $2h = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$

Multiply both sides by 4: $8h = \pi h^2 \cdot \frac{dh}{dt}$

For $h \neq 0$, divide by πh^2: $\frac{8h}{\pi h^2} = \frac{dh}{dt}$

So $\frac{dh}{dt} = \frac{8}{\pi h}$

When $h = 4$: $\frac{dh}{dt} = \frac{8}{\pi \cdot 4} = \frac{2}{\pi}$ m/min

Therefore, $\frac{dh}{dt} = \frac{2}{\pi}$ m/min