Case Study 2 : Surface Area and Volume

Case Study Paragraph:

A city garden commission plans a centre-piece fountain for a school campus. The fountain design consists of a solid cylindrical pedestal on the ground which supports a shallow frustum-shaped basin. A small hemispherical ornament sits at the centre of the basin as a decorative finial. The pedestal has radius 2 m and height 1.5 m. The frustum basin has lower radius 3 m, upper radius 1 m and vertical height 0.8 m. The hemisphere has radius 0.5 m. The contractor must estimate the exterior surface area to be painted (the base of the pedestal that rests on the ground is not painted) and compute the water capacity of the basin when filled to its brim. The basin is open at the top; only the exposed external surfaces are to be painted (pedestal curved surface, lateral surface of frustum, the top annulus of the basin and the curved surface of the hemisphere). Use π in symbolic answers and give numerical values correct to three significant figures where requested. This problem tests combining curved surface areas, frustum slant computations and volume of frustum.

MCQ Questions:

1. The curved surface area (CSA) of the cylindrical pedestal is:

   (a) $2\pi \times 2 \times 1.5 = 6\pi \text{ m}^2$
   (b) $2\pi \times 3 \times 1.5 = 9\pi \text{ m}^2$
   (c) $2\pi \times 2 \times 3 = 12\pi \text{ m}^2$
   (d) $\pi \times 2^2 = 4\pi \text{ m}^2$

   Answer: (a)
   Solution: CSA of a right circular cylinder $= 2\pi rh$. Here $r = 2\,\text{m}$, $h = 1.5\,\text{m}$ so CSA $= 2\pi \times 2 \times 1.5 = 6\pi \text{ m}^2 \approx 18.85 \text{ m}^2$.

2. The slant height l of the frustum basin (with $R = 3\,\text{m}$, $r = 1\,\text{m}$, $h = 0.8\,\text{m}$) is:

   (a) $\sqrt{(3-1)^2 + 0.8^2} = \sqrt{4 + 0.64}$
   (b) $\sqrt{(3+1)^2 + 0.8^2} = \sqrt{16 + 0.64}$
   (c) $\sqrt{3^2 + 1^2}$
   (d) 0.8

   Answer: (a)
   Solution: For a frustum, slant height $l = \sqrt{(R-r)^2 + h^2} = \sqrt{(3-1)^2 + 0.8^2} = \sqrt{4 + 0.64} = \sqrt{4.64} \approx 2.154$ m.

3. The lateral surface area (LSA) of the frustum basin is (use l from previous):

   (a) $\pi(R+r)l = \pi(3+1) \times 2.154 \text{ m}^2 \approx 27.07 \text{ m}^2$
   (b) $\pi(R-r)l = \pi(3-1) \times 2.154 \text{ m}^2 \approx 13.54 \text{ m}^2$
   (c) $2\pi Rh = 2\pi \times 3 \times 0.8 \text{ m}^2 \approx 15.08 \text{ m}^2$
   (d) $\pi r^2 = \pi \times 1^2 \approx 3.142 \text{ m}^2$

   Answer: (a)
   Solution: Lateral surface area of a frustum $= \pi(R+r)l$. With $R = 3$, $r = 1$ and $l \approx 2.154$, LSA $= \pi \times 4 \times 2.154 \approx 27.07 \text{ m}^2$.

4. The external horizontal area at the basin top (the top annulus) which is exposed and to be painted equals:

(a) $\pi(R^2 - r^2) = \pi(9 - 1) = 8\pi \text{ m}^2$

(b) $\pi r^2 = \pi \text{ m}^2$

(c) $\pi R^2 = 9\pi \text{ m}^2$

(d) $2\pi Rr = 6\pi \text{ m}^2$

Answer: (a)
Solution: The exposed top horizontal ring (annulus) area $= \pi(R^2 - r^2) = \pi(3^2 - 1^2) = 8\pi \text{ m}^2 \approx 25.13 \text{ m}^2$.

5. The total exterior area to be painted (curved surface of pedestal + lateral frustum + top annulus + hemisphere curved surface) is approximately:

(a) $6\pi + \pi(R + r)l + 8\pi + 2\pi(0.5)^2 \approx 72.6 \text{ m}^2$

(b) $6\pi + \pi(R + r)l + 8\pi - 2\pi(0.5)^2 \approx 71.0 \text{ m}^2$

(c) $\pi(R^2 + r^2) + 2\pi(0.5)^2 \approx 40.0 \text{ m}^2$

(d) $\pi(6 + 4 + 0.5) \approx 32.0 \text{ m}^2$

Answer: (a)
Solution: Compute each contribution:

$$\begin{aligned} \text{CSA of pedestal} &= 6\pi \text{ m}^2 \approx 18.85, \\ \text{LSA of frustum} &= \pi(R + r)l \approx 27.07, \\ \text{Top annulus} &= 8\pi \approx 25.13, \\ \text{CSA of hemisphere} &= 2\pi(0.5)^2 = 0.5\pi \approx 1.571. \end{aligned}$$

Summing: $6\pi + \pi(R + r)l + 8\pi + 0.5\pi \approx 72.62 \text{ m}^2$. Thus option (a) is correct.

Note: The following question concerns capacity.

6. The volume of water the frustum basin can hold when filled to the brim equals:

(a) $\frac{1}{3}\pi h(R^2 + r^2 + Rr) \approx 10.89 \text{ m}^3$

(b) $\pi r^2 h \approx 2.51 \text{ m}^3$

(c) $\frac{2}{3}\pi R^3 \approx 56.55 \text{ m}^3$

(d) $\frac{4}{3}\pi(3^3 - 1^3) \approx 110.0 \text{ m}^3$

Answer: (a)
Solution: Volume of a frustum $= \frac{1}{3}\pi h(R^2 + r^2 + Rr)$. Substitute $h = 0.8$, $R = 3$, $r = 1$:

$$V = \frac{1}{3}\pi \times 0.8 \times (9 + 1 + 3) = \frac{0.8\pi}{3} \times 13 \approx 10.891 \text{ m}^3.$$

Hence option (a) is correct.