Case Study:3

Ritika is designing a logo for a cultural event, and she wants the logo to represent a leaf-like symmetric shape. She models one side of the leaf using the curve $y = \sqrt{1-x^2}$, which is a semi-circle of radius 1 centered at the origin. She wants to calculate the exact area of this semi-circular region bounded above by the curve and below by the x-axis. Ritika needs the area to decide on the cost of etching this design using laser cutting. Using integral calculus, she knows that the area under a curve can be computed by a definite integral over a given interval. In this problem, she uses her understanding of symmetry and definite integration to find the desired area.

Concepts and Formulae Used:

- Area under a curve:

$$\text{Area} = \int_a^b f(x)\,dx$$

- Symmetry: The function $y = \sqrt{1-x^2}$ is even, so:

$$\int_{-1}^{1} f(x)\,dx = 2\int_0^1 f(x)\,dx$$

- This curve represents the upper half of a circle of radius 1.

MCQ Questions:

1. What type of curve is represented by $y = \sqrt{1-x^2}$?

   (a) Line

   (b) Parabola

   (c) Semicircle

   (d) Hyperbola

   Answer: (c)
   Solution: $y = \sqrt{1-x^2}$ is the upper half of the circle $x^2 + y^2 = 1$

2. What are the limits of integration for the area under the curve from left to right?

   (a) $-\infty$ to ∞

   (b) -1 to 1

   (c) 0 to 1

   (d) -1 to 0

   Answer: (b)
   Solution: The domain of $\sqrt{1-x^2}$ is from -1 to 1

3. Which of the following expressions represents the area under the curve $y = \sqrt{1-x^2}$ from $x = -1$ to $x = 1$?

(a) $\int_0^1 \sqrt{1-x^2}\,dx$

(b) $\int_{-1}^1 \sqrt{1-x^2}\,dx$

(c) $2\int_{-1}^0 \sqrt{1-x^2}\,dx$

(d) $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Answer: (b)
Solution: Area is given directly by $\int_{-1}^1 \sqrt{1-x^2}\,dx$

4. What is the value of $\int_{-1}^1 \sqrt{1-x^2}\,dx$?

   (a) π

   (b) $\frac{\pi}{2}$

   (c) $\frac{\pi}{4}$

   (d) $\frac{\pi}{3}$

   Answer: (b)
   Solution: The area under $y=\sqrt{1-x^2}$ from -1 to 1 is the area of a semicircle of radius 1:

$$\text{Area} = \frac{1}{2}\pi r^2 = \frac{\pi}{2}$$

5. How does symmetry help in evaluating $\int_{-1}^1 \sqrt{1-x^2}\,dx$?

   (a) It avoids integration

   (b) It allows us to double the integral over [0, 1]

   (c) It changes the integrand

   (d) It reduces the function to zero

   Answer: (b)
   Solution: Since the function is even, the area from -1 to 1 is:

$$\int_{-1}^1 f(x)\,dx = 2\int_0^1 f(x)\,dx$$