Case Study 2: Estimating the Height of a Tree from Two Observations

A group of Class 10 students are asked to estimate the height of a tall tree in the school playground without climbing it. They choose two points, A and B, on the same straight line from the base of the tree, with B closer to the tree than A. The students measure the horizontal distance between A and B, and using a clinometer they record the angles of elevation to the top of the tree from these two points. From this information they can form two right-angled triangles which share the same vertical height (the tree's height). By applying trigonometric ratios and simple algebra (eliminating the unknown horizontal distance to the tree), they derive an exact formula for the tree's height in terms of the known separation and the two measured angles. This method is practical, safe and commonly used in surveying and fieldwork. The exercise develops algebraic manipulation with trigonometric functions and reinforces understanding of tan as the ratio of the opposite side (height) to the adjacent side (horizontal distance).

Key relations and derived formulas:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}, \qquad \text{if } h \text{ is height, and distances from base are } x \text{ and } x-d,$$

then

$$h = x\tan\alpha, \qquad h = (x-d)\tan\beta.$$

Eliminating x gives the standard formula (when both observations lie on the same side of the object and A is farther):

$$\boxed{h \;=\; d\,\frac{\tan\alpha\,\tan\beta}{\tan\beta-\tan\alpha}}$$

(When observers stand on opposite sides at separation d and measure α, β to the top, the height is)

$$\boxed{h \;=\; d\,\frac{\tan\alpha\,\tan\beta}{\tan\alpha+\tan\beta}}.$$

MCQ Questions

1. Two students measure the angle of elevation to the top of a tree as follows: from point A (farther) $\alpha = 30^\circ$, and after walking 5 m towards the tree to point B (closer) they measure $\beta = 45^\circ$. The height h of the tree is:
   (a) $\frac{5}{2}(\sqrt{3}+1)$ m (b) $5\sqrt{3}$ m (c) 10 m (d) $\frac{5}{2}(\sqrt{3}-1)$ m

   Answer: (a) $\frac{5}{2}(\sqrt{3}+1)$ m.

   Solution: Using
   $$h = d\frac{\tan\alpha\tan\beta}{\tan\beta-\tan\alpha},$$
   with $\tan 30^\circ = \frac{1}{\sqrt{3}}$, $\tan 45^\circ = 1$, $d = 5$:
   $$h = 5\cdot\frac{\frac{1}{\sqrt{3}}\cdot 1}{1-\frac{1}{\sqrt{3}}} = \frac{5}{\sqrt{3}-1} = \frac{5(\sqrt{3}+1)}{2} = \frac{5}{2}(\sqrt{3}+1).$$

   Numerical check: ≈ 6.83 m.

2. From a point at distance 20 m from a tower the angle of elevation is 60°. The height of the tower is:
(a) 20 m (b) $20\sqrt{3}$ m (c) $10\sqrt{3}$ m (d) 40 m

Answer: (b) $20\sqrt{3}$ m.

Solution: For a single observation, $h = (\text{distance}) \cdot \tan\theta$. Since $\tan 60^\circ = \sqrt{3}$,

$$h = 20 \cdot \sqrt{3} = 20\sqrt{3}.$$

3. An observer at point P measures angle $\alpha = 30^\circ$ to the top of a pole. He walks 15 m towards the pole to point Q and now measures $\beta = 60^\circ$. The horizontal distance from the original point P to the base of the pole equals:
(a) 15 m (b) 22.5 m (c) 30 m (d) 45 m

Answer: (b) 22.5 m.

Solution: From derivation,

$$x = d\frac{\tan\beta}{\tan\beta - \tan\alpha},$$

where x is original distance from P to base, $d = 15$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$, $\tan 60^\circ = \sqrt{3}$. Thus

$$x = 15 \cdot \frac{\sqrt{3}}{\sqrt{3} - \frac{1}{\sqrt{3}}} = 15 \cdot \frac{\sqrt{3}}{\frac{3-1}{\sqrt{3}}} = 15 \cdot \frac{3}{2} = 22.5 \text{ m}.$$

4. Two observers stand on opposite sides of a tower with separation 30 m. Their angles of elevation to the top are $\alpha = 45^\circ$ and $\beta = 30^\circ$. The height h of the tower is:
(a) $15(\sqrt{3}+1)$ m (b) $15(\sqrt{3}-1)$ m (c) $10\sqrt{3}$ m (d) 30 m

Answer: (b) $15(\sqrt{3}-1)$ m.

Solution: For observers on opposite sides,

$$h = d\frac{\tan\alpha\tan\beta}{\tan\alpha + \tan\beta}.$$

Here $d = 30$, $\tan 45^\circ = 1$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$. So

$$h = 30 \cdot \frac{1 \cdot \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = 30 \cdot \frac{1}{\sqrt{3}+1} = 15(\sqrt{3}-1).$$

Numerical check: ≈ 10.98 m.

5. A lamp post casts a shadow of length 16 m when the angle of elevation of the Sun is $\theta = \arctan(\frac{3}{4})$. The height of the lamp post is:
(a) 9 m (b) 12 m (c) 10 m (d) 16 m

Answer: (b) 12 m.

Solution: If $\tan\theta = \frac{3}{4}$ and shadow length = 16 m, then

$$h = 16 \cdot \tan\theta = 16 \cdot \frac{3}{4} = 12 \text{ m}.$$