Case Study 2:

A horticulture club is redesigning a recreational circular lake and its surrounds to make the area educational for Grade 10 students learning about areas related to circles. The original lake is a perfect circle of radius 21 m. Around the lake there is a concentric circular walking belt (an annular strip) used for jogging and exhibitions. At one sector of the outer edge (a $120°$ sector) the club intends to lay decorative stones forming a fan-shaped pattern (a sector of the outer circle). On the opposite side of the lake, at a radius of 14 m from the centre, a narrow service path (a straight chord) runs and cuts off a small circular segment of the inner garden; the perpendicular distance from the centre to the chord is 10 m. Additionally, the club will convert a ring-sector between radii 18 m and 21 m subtending an angle of $150°$ into a flowerbed. The students must compute circumferences, areas of circles, annuli, sectors, segments and ring-sectors to prepare accurate material quantities and labels for the information boards. The paragraph above describes the exact geometric layout; all questions that follow are based only on this description and require exact or simplified symbolic answers where appropriate.

MCQ Questions:

1. What is the circumference of the lake (in metres)?

   (a) 21π
   (b) 42π
   (c) 84π
   (d) 441π

   Answer: (b) 42π.
   Solution: Circumference $= 2\pi r = 2\pi(21) = 42\pi$ m.

2. Find the area of the annular walking belt that lies between the lake (radius 21 m) and a concentric inner ornamental moat of radius 7 m which the club plans to keep water-free.

   (a) 392π m^2
   (b) 350π m^2
   (c) 1225π m^2
   (d) 98π m^2

   Answer: (a) 392π m^2.
   Solution: Area of annulus $= \pi(R^2 - r^2) = \pi(21^2 - 7^2) = \pi(441 - 49) = 392\pi$ m^2.

3. The decorative fan-shaped stone pattern occupies a $120°$ sector of the outer circle of radius 21 m. What is the area of that sector?

   (a) 147π m^2
   (b) 77π m^2
   (c) 63π m^2
   (d) 441π m^2

Answer: (a) 147π m^2.

Solution: Sector area $= \dfrac{\theta}{360^\circ}\pi r^2$ with $\theta = 120^\circ$, $r = 21$. Thus area $= \dfrac{120}{360}\pi(21^2) = \dfrac{1}{3}\pi \cdot 441 = 147\pi$ m^2.

4. On the inner garden (radius 14 m) the service chord is at perpendicular distance 10 m from the centre and cuts off the *smaller* circular segment. Express the area of this smaller segment exactly (in simplest symbolic form) and choose the correct option.

   (a) $196\arccos(\frac{5}{7}) - 40\sqrt{6}$ m^2

   (b) $196\arccos(\frac{5}{7}) - 20\sqrt{6}$ m^2

   (c) $98\arccos(\frac{5}{7}) - 40\sqrt{6}$ m^2

   (d) $196\arccos(\frac{10}{14}) - 40\sqrt{6}$ m^2

   Answer: (a) $196\arccos(\frac{5}{7}) - 40\sqrt{6}$ m^2.

   Solution: Let the radius be $r = 14$ and let the perpendicular distance from centre to chord be $d = 10$. Denote $\alpha = \arccos(\frac{d}{r}) = \arccos(\frac{5}{7})$. The central angle corresponding to the smaller segment is 2α. The area of a segment with central angle 2α is

   $$\text{segment area} = \tfrac{1}{2}r^2(2\alpha - \sin 2\alpha) = r^2(\alpha - \sin\alpha\cos\alpha).$$

   Compute $\cos\alpha = \frac{5}{7}$ and $\sin\alpha = \sqrt{1 - (\frac{5}{7})^2} = \frac{2\sqrt{6}}{7}$. Hence

   $$\sin\alpha\cos\alpha = \frac{5}{7}\cdot\frac{2\sqrt{6}}{7} = \frac{10\sqrt{6}}{49}.$$

   Therefore the segment area equals

   $$r^2\left(\alpha - \frac{10\sqrt{6}}{49}\right) = 14^2\alpha - 14^2\cdot\frac{10\sqrt{6}}{49} = 196\arccos(\tfrac{5}{7}) - 40\sqrt{6}\ \text{m}^2,$$

   which matches option (a). Note option (d) is equivalent to (a) in the arccos argument but (a) is the simplified correct form.

5. The flowerbed occupies the ring-sector between radii 18 m and 21 m subtending an angle of 150°. What is the exact area of this ring-sector?

   (a) $\dfrac{195}{4}\pi$ m^2

   (b) 48π m^2

   (c) $\dfrac{117}{6}\pi$ m^2

   (d) 50π m^2

   Answer: (a) $\dfrac{195}{4}\pi$ m^2.

   Solution: Area of a ring-sector $= \dfrac{\theta}{360^\circ}\pi(R^2 - r^2)$ with $\theta = 150^\circ$, $R = 21$, $r = 18$. Compute $R^2 - r^2 = 441 - 324 = 117$. Thus area $= \dfrac{150}{360}\pi\cdot 117 = \dfrac{5}{12}\cdot 117\pi = \dfrac{585}{12}\pi = \dfrac{195}{4}\pi$ m^2.