Case Study 3: Production Line with an Increasing Daily Output

A small workshop manufactures handcrafted lamps. On the first working day of a new month the team produces 13 lamps. Because of improved coordination and slight process improvements, each subsequent working day they produce 5 more lamps than the previous day. Thus the daily production (in lamps) forms an arithmetic progression. The production manager wants to analyse short-term and medium-term outputs: how many lamps are produced on a given day, how many lamps are produced in the first several days, how many days are needed to reach a fixed production target, and what is the total production between two given days. Use the theory of arithmetic progressions (nth term and sum of the first n terms) to answer the questions below. This case mixes routine use of formulas with quadratic solving and difference-of-sums computations that are typical of intermediate-to-difficult Grade 10 problems.

Basic formulas and properties used:

$$\text{General AP: } a,\ a+d,\ a+2d,\ a+3d,\ldots$$
$$n\text{th term: } a_n = a+(n-1)d.$$
$$\text{Sum of first } n \text{ terms: } S_n = \frac{n}{2}(2a+(n-1)d) = \frac{n}{2}(a_1+a_n).$$
$$\text{Sum of terms from } r \text{ to } s \text{ (with } 1 \le r \le s\text{): } \sum_{k=r}^{s} a_k = S_s - S_{r-1}.$$

In this case $a = 13$ and $d = 5$, so $a_n = 13 + 5(n-1) = 5n + 8$ and

$$S_n = \frac{n}{2}(2\cdot 13 + (n-1)\cdot 5) = \frac{n(5n+21)}{2}.$$

MCQ Questions

1. What is the common difference d of the daily production AP?
   (a) 3 (b) 4 (c) 5 (d) 6

   Answer: (c) 5.

   Solution: The production increases by 5 lamps each day. Thus $d = 5$.

2. How many lamps are produced on the 15th working day?
   (a) 78 (b) 79 (c) 82 (d) 83

   Answer: (d) 83.

   Solution: Use $a_n = 5n + 8$. For $n = 15$,

   $$a_{15} = 5(15) + 8 = 75 + 8 = 83.$$

3. After how many working days will the workshop have produced exactly 486 lamps in total?
   (a) 10 (b) 11 (c) 12 (d) 13

   Answer: (c) 12.

   Solution: We need $S_n = 486$. Using $S_n = \frac{n(5n+21)}{2}$,

   $$\frac{n(5n+21)}{2} = 486 \quad \Rightarrow \quad n(5n+21) = 972.$$

   So

   $$5n^2 + 21n - 972 = 0.$$

Solve the quadratic. Compute discriminant:

$$\Delta = 21^2 + 4 \cdot 5 \cdot 972 = 441 + 19440 = 19881.$$

Now $\sqrt{19881} = 141$ because $141^2 = 19881$. Thus

$$n = \frac{-21 \pm 141}{10}.$$

Positive root:

$$n = \frac{-21 + 141}{10} = \frac{120}{10} = 12.$$

Hence after 12 days total production is 486.

4. What is the total number of lamps produced from the 5th to the 15th working day inclusive?
(a) 620 (b) 638 (c) 656 (d) 674

Answer: (b) 638.

Solution: Sum from $k = 5$ to 15 equals $S_{15} - S_4$. Compute S_{15}:

$$S_{15} = \frac{15(5 \cdot 15 + 21)}{2} = \frac{15(75 + 21)}{2} = \frac{15 \cdot 96}{2} = 15 \cdot 48 = 720.$$

Compute S_4:

$$S_4 = \frac{4(5 \cdot 4 + 21)}{2} = \frac{4(20 + 21)}{2} = \frac{4 \cdot 41}{2} = 2 \cdot 41 = 82.$$

Hence total $= 720 - 82 = 638$.

5. For how many initial working days n is the average daily production (average of the first n terms) equal to 48 lamps?
(a) 10 (b) 12 (c) 15 (d) 18

Answer: (c) 15.

Solution: Average of first n terms is $\frac{S_n}{n}$. Using $S_n = \frac{n(5n+21)}{2}$,

$$\frac{S_n}{n} = \frac{5n + 21}{2}.$$

Set this equal to 48:

$$\frac{5n + 21}{2} = 48 \quad \Rightarrow \quad 5n + 21 = 96 \quad \Rightarrow \quad 5n = 75$$

so $n = 15$. Thus after the first 15 days the average daily production is 48 lamps.