Case Study 3:

A city council plans a circular open-air amphitheatre for community events. The outer boundary of the amphitheatre is a perfect circle of radius 28 m. At the centre there is a decorative circular pond of radius 8 m used as a focal feature. The seating area occupies the annular region between the pond and the outer boundary. For an annual festival the council reserves a VIP zone which is a $135°$ sector of the outer circle (seating included) and marks it with special tiles. A service walkway runs across the seating area and is designed as a straight chord of the outer circle at a perpendicular distance 20 m from the centre; this chord cuts off a smaller circular segment that will be fenced for equipment. Near the outer rim, a decorative mosaic will occupy a ring-sector between radii 24 m and 28 m subtending an angle of $60°$. The engineering team must compute exact and simplified areas and lengths (in terms of π where appropriate) for materials, fencing and tiles. All numerical answers must be exact symbolic expressions or simplified fractions involving π and radicals as required.

MCQ Questions:

1. What is the circumference of the amphitheatre's outer boundary?

   (a) 56π m
   (b) 28π m
   (c) 112π m
   (d) 784π m

   Answer: (a) 56π m.
   Solution: Circumference $= 2\pi R$ with $R = 28$. Compute step by step:

   $$2 \times 28 = 56, \qquad \text{so } 2\pi R = 56\pi.$$

2. What is the exact area of the seating annulus (area between the outer boundary and the pond)?

   (a) 720π m^2
   (b) 784π m^2
   (c) 640π m^2
   (d) 576π m^2

   Answer: (a) 720π m^2.
   Solution: Area of annulus $= \pi(R^2 - r^2)$ with $R = 28$, $r = 8$. Compute $R^2 = 28^2 = 784$, $r^2 = 8^2 = 64$. Then

   $$R^2 - r^2 = 784 - 64 = 720,$$

   hence area $= 720\pi$ m^2.

3. The VIP zone is a $135°$ sector of the outer circle. What is its area?

   (a) 294π m^2
   (b) 196π m^2
   (c) 392π m^2

(d) 245π m^2

Answer: (a) 294π m^2.

Solution: Sector area $= \dfrac{\theta}{360^\circ}\pi R^2$ with $\theta = 135^\circ$, $R = 28$. Compute:

$$\frac{135}{360} = \frac{3}{8}, \qquad R^2 = 784,$$

thus sector area $= \dfrac{3}{8} \cdot 784\pi$. Compute $784 \nabla \cdot 8 = 98$, then $98 \times 3 = 294$, so area $= 294\pi$ m^2.

4. The service chord is at perpendicular distance $d = 20$ m from the centre of the outer circle ($R = 28$ m). The chord cuts off the smaller circular segment. Which expression equals the exact area of that smaller segment?

   (a) $784 \arccos(\frac{5}{7}) - 160\sqrt{6}$ m^2
   (b) $392 \arccos(\frac{5}{7}) - 80\sqrt{6}$ m^2
   (c) $784 \arccos(\frac{20}{28}) - 80\sqrt{6}$ m^2
   (d) $392 \arccos(\frac{20}{28}) - 160\sqrt{6}$ m^2

   Answer: (a) $784 \arccos(\frac{5}{7}) - 160\sqrt{6}$ m^2.
   Solution: Let $R = 28$, $d = 20$. Set $\theta = \arccos(\frac{d}{R}) = \arccos(\frac{20}{28}) = \arccos(\frac{5}{7})$. The area of the smaller segment is

   $$\text{segment area} = R^2\theta - d\sqrt{R^2 - d^2}.$$

   Compute $R^2 = 28^2 = 784$. Next $R^2 - d^2 = 784 - 400 = 384$. Factor: $384 = 64 \cdot 6$, so $\sqrt{384} = 8\sqrt{6}$. Then

   $$d\sqrt{R^2 - d^2} = 20 \cdot 8\sqrt{6} = 160\sqrt{6}.$$

   Therefore the area equals

   $$784 \arccos(\tfrac{5}{7}) - 160\sqrt{6}\ \text{m}^2,$$

   which matches option (a). (Options (c) and (d) use equivalent arccos arguments but their coefficients on the radical term are incorrect; options (b) and (d) have incorrect overall factors.)

5. The decorative mosaic occupies the ring-sector between radii 24 m and 28 m subtending an angle 60°. What is its exact area?

   (a) $\dfrac{104}{3}\pi$ m^2
   (b) $\dfrac{208}{3}\pi$ m^2
   (c) 48π m^2
   (d) $\dfrac{52}{3}\pi$ m^2

   Answer: (a) $\dfrac{104}{3}\pi$ m^2.

   Solution: Ring-sector area $= \dfrac{\theta}{360^\circ}\pi(R^2 - r^2)$ with $\theta = 60^\circ$, $R = 28$, $r = 24$. Compute $R^2 - r^2 = 28^2 - 24^2 = 784 - 576 = 208$. Next $\dfrac{60}{360} = \dfrac{1}{6}$. Thus area $= \dfrac{1}{6} \cdot 208\pi = \dfrac{208}{6}\pi$. Simplify $208 \nabla \cdot 2 = 104$, $6 \nabla \cdot 2 = 3$, hence area $= \dfrac{104}{3}\pi$ m^2.