Case Study 5

In three-dimensional space, a plane can be described in multiple forms. The general Cartesian form is given as:

$$ax + by + cz + d = 0$$

where a, b, c are the direction ratios of the normal to the plane, and d is a constant. The vector form is:

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

where $\vec{r}$ is the position vector of a variable point on the plane, $\vec{a}$ is a known point on the plane, and $\vec{n}$ is a normal vector to the plane.

Important applications include finding the angle between two planes, checking if a point lies on a plane, and computing the perpendicular distance from a point to a plane. The normal form of a plane is useful when the perpendicular from the origin to the plane is known:

$$x \cos\alpha + y \cos\beta + z \cos\gamma = p$$

where α, β, γ are the angles made with coordinate axes and p is the length of perpendicular from origin to the plane.

MCQ Questions:

1. What is the Cartesian equation of the plane passing through point $(1, 2, 3)$ and perpendicular to the vector $\vec{n} = 2\hat{i} - \hat{j} + 4\hat{k}$?

   (a) $2x - y + 4z = 0$
   (b) $2x - y + 4z = 11$
   (c) $x + 2y + 3z = 0$
   (d) $2x + y - 4z = 11$

   Answer: (b)

   Solution: Using point $(x_0, y_0, z_0) = (1, 2, 3)$ and normal vector $\vec{n} = (2, -1, 4)$:

   $$2(x - 1) - (y - 2) + 4(z - 3) = 0 \Rightarrow 2x - y + 4z = 11$$

2. Find the distance of the point $(1, 1, 1)$ from the plane $2x - 3y + 6z - 2 = 0$.

   (a) $\frac{5}{7}$
   (b) $\frac{3}{7}$
   (c) $\frac{1}{7}$
   (d) $\frac{2}{7}$

   Answer: (a)

   Solution: Use formula:

   $$\text{Distance} = \left| \frac{2(1) - 3(1) + 6(1) - 2}{\sqrt{2^2 + (-3)^2 + 6^2}} \right| = \left| \frac{3}{7} \right| = \frac{3}{7}$$

   Correction: Correct answer is (b)

3. Which of the following is the normal form of a plane?

   (a) $ax + by + cz = 0$
   (b) $x \cos \alpha + y \cos \beta + z \cos \gamma = p$
   (c) $ax + by + cz + d = 0$
   (d) $x^2 + y^2 + z^2 = R^2$

   Answer: (b)
   Solution: The normal form is:

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p$$

   where p is the perpendicular from the origin, and $\cos \alpha, \cos \beta, \cos \gamma$ are the direction cosines of the normal.

4. Find the vector equation of the plane passing through point $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and normal vector $\vec{n} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.

   (a) $\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) = 3$
   (b) $\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) = 0$
   (c) $\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) = 9$
   (d) $\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) = 11$

   Answer: (d)
   Solution: Plane equation:

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} = (3)(1) + (2)(2) + (-1)(2) = 3 + 4 - 2 = 5$$

   Correction: Correct RHS is 5. None of the options are correct. Replace option (a) with:

$$\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) = 5$$

5. The angle θ between the planes $x - 2y + 3z = 4$ and $3x + y - z = 5$ is given by:

   (a) $\cos \theta = \frac{-2}{\sqrt{14} \cdot \sqrt{11}}$
   (b) $\cos \theta = \frac{2}{\sqrt{14} \cdot \sqrt{11}}$
   (c) $\cos \theta = \frac{0}{\sqrt{14} \cdot \sqrt{11}}$
   (d) $\cos \theta = \frac{1}{\sqrt{14} \cdot \sqrt{11}}$

   Answer: (b)
   Solution: Normal vectors:

$$\vec{n}_1 = (1, -2, 3), \quad \vec{n}_2 = (3, 1, -1)$$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{1 \cdot 3 + (-2) \cdot 1 + 3 \cdot (-1)}{\sqrt{1^2 + 4 + 9} \cdot \sqrt{9 + 1 + 1}} = \frac{3 - 2 - 3}{\sqrt{14} \cdot \sqrt{11}} = \frac{-2}{\sqrt{154}}$$

   Correction: Correct answer is (a)