Case Study: 2

An engineer is designing a water channel in the shape of a region bounded by the curves $y = \sqrt{x}$ and $y = x^2$. To estimate the capacity of the channel, she needs to compute the cross-sectional area between these two curves over a specific interval. These functions intersect at two points which define the limits for computing the enclosed area. The area between these curves will help her determine the volume of water the channel can carry per unit length. She turns to integral calculus to evaluate the region precisely, using the formula for the area between two curves.

Concepts and Formulae Used:

- Points of intersection determine the limits of integration.
- If $f(x) \geq g(x)$, then area is:

$$\text{Area} = \int_a^b [f(x) - g(x)]\, dx$$

MCQ Questions:

1. What are the points of intersection of $y = \sqrt{x}$ and $y = x^2$?

   (a) $x = 0$ and $x = 2$

   (b) $x = 0$ and $x = 1$

   (c) $x = -1$ and $x = 1$

   (d) $x = 1$ and $x = 4$

   Answer: (b)
   Solution: Set $\sqrt{x} = x^2 \Rightarrow x = 0, x = 1$

2. Which function lies above the other in the interval $[0, 1]$?

   (a) $y = x^2$ lies above $y = \sqrt{x}$

   (b) Both are equal

   (c) $y = \sqrt{x}$ lies above $y = x^2$

   (d) Cannot determine

   Answer: (c)
   Solution: For $0 < x < 1$, $\sqrt{x} > x^2$

3. What is the correct expression for the area between the curves?

   (a) $\int_0^1 (\sqrt{x} - x^2)\, dx$

   (b) $\int_0^1 (x^2 - \sqrt{x})\, dx$

(c) $\int_0^1 (x^2 + \sqrt{x})\, dx$

(d) $\int_0^1 (\sqrt{x} + x^2)\, dx$

Answer: (a)
Solution: The area is given by $\int_0^1 [\sqrt{x} - x^2]\, dx$

4. Evaluate the integral $\int_0^1 \sqrt{x}\, dx$

(a) $\frac{2}{3}$

(b) $\frac{1}{2}$

(c) $\frac{3}{2}$

(d) 1

Answer: (a)
Solution: $\int_0^1 x^{1/2}\, dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}$

5. Find the total area enclosed between the curves $y = \sqrt{x}$ and $y = x^2$ from $x = 0$ to $x = 1$

(a) $\frac{1}{6}$

(b) $\frac{2}{3}$

(c) $\frac{1}{3}$

(d) $\frac{5}{6}$

Answer: (d)
Solution:

$$\int_0^1 (\sqrt{x} - x^2)\, dx = \int_0^1 x^{1/2}\, dx - \int_0^1 x^2\, dx = \frac{2}{3} - \frac{1}{3} = \frac{5}{6}$$