Case Study 3: Resolving Vectors into Components

In a space research lab, engineers are developing a satellite launch simulation model. To program the movement of the satellite in 3D space, they need to calculate the position and direction of motion at various stages. Every motion is resolved into components along the three coordinate axes. They study how vectors can be expressed in terms of direction cosines, direction ratios, and unit vectors. By using the position vector, they determine the precise location of the satellite at any instant. Direction cosines help in analyzing the orientation of movement in space. All the engineers must be proficient in converting vectors from geometric to component form in both 2D and 3D, which is essential for accurate navigation.

Theory and Formulae Related to Vector Components:

- Resolution in 3D: $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
- Magnitude: $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- Direction cosines: $\cos\alpha = \frac{a_1}{|\vec{a}|}, \cos\beta = \frac{a_2}{|\vec{a}|}, \cos\gamma = \frac{a_3}{|\vec{a}|}$
- Property: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
- Position vector of a point $P(x, y, z)$ is $\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

MCQ Questions

1. What is the position vector of the point $P(3, -2, 5)$?

   (a) $3\hat{i} - 2\hat{j} + 5\hat{k}$

   (b) $-3\hat{i} + 2\hat{j} + 5\hat{k}$

   (c) $3\hat{i} + 2\hat{j} - 5\hat{k}$

   (d) $2\hat{i} + 3\hat{j} + 5\hat{k}$

   Answer: (a)
   Solution: The position vector of $P(x, y, z)$ is $x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow 3\hat{i} - 2\hat{j} + 5\hat{k}$

2. If $\vec{a} = 4\hat{i} + 3\hat{j} + 12\hat{k}$, find its magnitude.

   (a) 13

   (b) $\sqrt{29}$

   (c) 11

   (d) $\sqrt{169}$

Answer: (a)
Solution: $|\vec{a}| = \sqrt{4^2 + 3^2 + 12^2} = \sqrt{16 + 9 + 144} = \sqrt{169} = 13$

3. If a vector makes angles α, β, γ with the coordinate axes and the direction cosines are $\cos\alpha = \frac{3}{13}, \cos\beta = \frac{4}{13}, \cos\gamma = \frac{12}{13}$, then verify the identity.

   (a) $\cos^2\alpha + \cos^2\beta = \cos^2\gamma$

   (b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

   (c) $\cos\alpha + \cos\beta + \cos\gamma = 0$

   (d) None of the above

   Answer: (b)
   Solution: $\left(\frac{3}{13}\right)^2 + \left(\frac{4}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = \frac{9+16+144}{169} = \frac{169}{169} = 1$

4. Find the direction cosines of vector $\vec{a} = 6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$

   (a) $\frac{6}{7}, \frac{-3}{7}, \frac{2}{7}$

   (b) $\frac{6}{\sqrt{49}}, \frac{3}{\sqrt{49}}, \frac{2}{\sqrt{49}}$

   (c) $\frac{3}{7}, \frac{2}{7}, \frac{6}{7}$

   (d) None of the above

   Answer: (a)
   Solution: Magnitude $= \sqrt{6^2 + (-3)^2 + 2^2} = \sqrt{36 + 9 + 4} = \sqrt{49} = 7$, so direction cosines $= \frac{6}{7}, \frac{-3}{7}, \frac{2}{7}$

5. Which of the following is correct about the direction cosines of any vector?

   (a) $\cos^2\alpha + \cos^2\beta = \cos^2\gamma$

   (b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 0$

   (c) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

   (d) None of these

   Answer: (c)
   Solution: By definition of direction cosines: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$