Case Study 3: Solving First-Order, First-Degree Differential Equations

A company is studying the temperature variation in a metal rod heated at one end and insulated at the other. The temperature y at a point x along the rod is governed by the differential equation:

$$\frac{dy}{dx} + Py = Q$$

This is a linear differential equation. Solving such equations involves finding the integrating factor (IF), which is given by:

$$\text{IF} = e^{\int P dx}$$

The general solution is:

$$y \cdot \text{IF} = \int Q \cdot \text{IF}\, dx + C$$

Other types of solvable first-order differential equations include:

- **Variables separable:** $\frac{dy}{dx} = f(x)g(y)$
- **Homogeneous:** $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, solved using substitution $y = vx$

Engineers often model heat flow, current in circuits, or population changes using these techniques.

MCQ Questions

1. The standard form of a linear differential equation is:

   (a) $\frac{dy}{dx} = f(x)g(y)$

   (b) $y' + Py = Q$

   (c) $y'' + y = 0$

   (d) $\frac{dy}{dx} = \frac{y}{x}$

   Answer: (b) $y' + Py = Q$

   Solution: A first-order linear differential equation has the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x.

2. The integrating factor for $\frac{dy}{dx} + 3y = 6$ is:

   (a) e^{3x}

   (b) e^{-3x}

   (c) $3x$

   (d) e^{6x}

   Answer: (a) e^{3x}

   Solution: Here, $P = 3$. So the integrating factor is:

$$IF = e^{\int 3dx} = e^{3x}$$

3. Solve the differential equation :

$$\frac{dy}{dx} + y\tan x = \sin x$$

Which one of the following is the general solution?

(a) $y = \cos x\,(\ln|\cos x| + C)$
(b) $y = \cos x\,(-\ln|\cos x| + C)$
(c) $y = \sin x \cos x + C\sec x$
(d) $y = \cos x\,(\ln|\sec x| + C)$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x) = \tan x$, and $Q(x) = \sin x$.

Step 1: Find the Integrating Factor (IF)

$$\text{IF} = e^{\int P(x)\,dx} = e^{\int \tan x\,dx} = e^{-\ln|\cos x|} = \frac{1}{\cos x}$$

Step 2: Multiply both sides by IF

$$\frac{1}{\cos x}\cdot\frac{dy}{dx} + \frac{y\tan x}{\cos x} = \frac{\sin x}{\cos x}$$

But instead, using the standard method:

$$\frac{d}{dx}\left(y\cdot\frac{1}{\cos x}\right) = \frac{\sin x}{\cos x}$$

$$\frac{d}{dx}\left(\frac{y}{\cos x}\right) = \tan x$$

Step 3: Integrate both sides

$$\int \frac{d}{dx}\left(\frac{y}{\cos x}\right)dx = \int \tan x\,dx \Rightarrow \frac{y}{\cos x} = -\ln|\cos x| + C$$

Step 4: Solve for y

$$y = \cos x\,(-\ln|\cos x| + C)$$

This is a valid general solution, but we can also check the given options to find the correct one by **differentiating** them.

Final Answer:

None of the given options match the correct general solution:

$$\boxed{y = \cos x\,(-\ln|\cos x| + C)}$$

4. The equation $\frac{dy}{dx} = \frac{x+y}{x}$ is:
   (a) Linear
   (b) Homogeneous
   (c) Variables separable
   (d) Not solvable

   Answer: (b) Homogeneous

   Solution: Rewrite as:
   $$\frac{dy}{dx} = 1 + \frac{y}{x} = F\left(\frac{y}{x}\right)$$
   Substituting $y = vx$ reduces it to a separable form. Hence it is homogeneous.

5. Which substitution is used to solve a homogeneous differential equation?
   (a) $y = vx$
   (b) $y = x + v$
   (c) $y = v^x$
   (d) $x = vy$

   Answer: (a) $y = vx$

   Solution: For a homogeneous equation $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, use $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.