Case Study 1: Introduction to Differential Equations and General/Particular Solutions

A pharmaceutical company is studying the change in concentration of a drug in the bloodstream. After a single injection, the rate at which the drug leaves the body is found to be proportional to the amount present at any time. Mathematically, this scenario is modeled using a first-order differential equation. If $y(t)$ represents the concentration at time t, then the equation takes the form:

$$\frac{dy}{dt} = -ky, \quad k > 0$$

This is a separable and linear differential equation. Its general solution is given by:

$$y(t) = Ae^{-kt}$$

where A is an arbitrary constant determined by initial conditions. The solution $y(t)$ is called a general solution, while assigning a value to A based on an initial concentration leads to the particular solution. Such models are also used in radioactive decay, cooling laws, and financial depreciation problems. Understanding the difference between general and particular solutions is crucial in interpreting the practical implications of a differential equation.

MCQ Questions

1. What is the order and degree of the differential equation $\frac{dy}{dt} = -ky$?

   (a) Order 2, Degree 1

   (b) Order 1, Degree 1

   (c) Order 1, Degree 2

   (d) Order 2, Degree 2

   Answer: (b) Order 1, Degree 1

   Solution: The highest order derivative is $\frac{dy}{dt}$, which makes it order 1. The power of the derivative is 1 (no square or higher power), hence degree is 1.

2. The general solution of $\frac{dy}{dt} = -ky$ is:

   (a) $y = Ae^{kt}$

   (b) $y = Ae^{-kt}$

   (c) $y = A + kt$

   (d) $y = A - kt$

   Answer: (b) $y = Ae^{-kt}$

   Solution: Separating variables and integrating:

$$\frac{dy}{y} = -kdt \Rightarrow \ln|y| = -kt + C \Rightarrow y = Ae^{-kt}$$

3. If the initial concentration is 50 mg and $k = 0.2$, what is the concentration after 5 hours?

   (a) 20.33 mg

   (b) 18.39 mg

   (c) 22.47 mg

   (d) 15.63 mg

   Answer: (b) 18.39 mg

   Solution:

$$y(t) = 50e^{-0.2\times5} = 50e^{-1} \approx 50 \times 0.3679 = 18.39 \text{ mg}$$

4. What distinguishes a particular solution from a general solution?

   (a) Particular solution contains arbitrary constants

   (b) Particular solution satisfies a given initial condition

   (c) General solution is more accurate

   (d) General solution is used only in applications

   Answer: (b) Particular solution satisfies a given initial condition

   Solution: The general solution contains arbitrary constants. When a specific condition (e.g., $y(0) = y_0$) is applied, the arbitrary constant is fixed, giving the particular solution.

5. Which of the following is a valid integrating factor (IF) for the linear differential equation $\frac{dy}{dt} + ky = 0$?

   (a) e^{-kt}

   (b) e^{kt}

   (c) $e^{\frac{1}{k}}$

   (d) kt

   Answer: (b) e^{kt}

   Solution: The equation is in standard linear form: $\frac{dy}{dt} + ky = 0$. The integrating factor is:

$$IF = e^{\int kdt} = e^{kt}$$