Case Study 3: Substitution Method and Simplification Using Properties

Aryan was solving definite integrals during a weekend study session. His teacher had recently explained the substitution method in class and also revised key properties such as linearity, symmetry, and splitting the interval. Aryan found some integrals easier when he made a proper substitution, such as $x = a - t$ or $x = \sin\theta$ in trigonometric integrals.

He was also taught how to apply the identity:

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

and the importance of recognizing whether a function is even or odd to use:

$$\text{If } f(x) \text{ is even, then } \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \quad \text{if odd, then } \int_{-a}^{a} f(x)\,dx = 0$$

Aryan's goal was to use substitution and properties to simplify integrals instead of expanding or directly integrating. Answer the following questions based on Aryan's approach.

MCQ Questions

1. Evaluate: $\int_0^{\frac{\pi}{2}} \sin x \cos x\,dx$

   (a) $\frac{1}{2}$
   (b) $\frac{1}{4}$
   (c) $\frac{\pi}{4}$
   (d) 1

   Answer: (1) $\frac{1}{2}$
   Solution: Use the identity: $\sin x \cos x = \frac{1}{2}\sin 2x$

$$\int_0^{\frac{\pi}{2}} \sin x \cos x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x\,dx = \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\cdot\left(-\frac{1}{2}(\cos\pi - \cos 0)\right)$$

$$= \frac{1}{2}\cdot\left(-\frac{1}{2}(-1-1)\right) = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

2. Evaluate using substitution: $\int_0^1 \frac{1}{1+x^2}dx$

   (a) $\frac{\pi}{4}$
   (b) 1
   (c) $\ln 2$
   (d) $\frac{\pi}{2}$

   Answer: (1) $\frac{\pi}{4}$
   Solution:

$$\int_0^1 \frac{1}{1+x^2}dx = \left[\tan^{-1} x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

3. Evaluate using identity: $\int_0^2 \frac{1}{x+1}dx$

   (a) $\ln 2$

(b) $\ln 3$

(c) $\ln 4$

(d) 1

Answer: (2) $\ln 3$
Solution:

$$\int_0^2 \frac{1}{x+1}dx = [\ln|x+1|]_0^2 = \ln 3 - \ln 1 = \ln 3$$

4. Evaluate using substitution: $\int_0^{\pi/2} \frac{1}{1+\sin x}\,dx$

   (a) π

   (b) 1

   (c) $\frac{\pi}{2}$

   (d) $\ln 2$

   Answer: (3) $\frac{\pi}{2}$
   Solution: Let $I = \int_0^{\pi/2} \frac{1}{1+\sin x}dx$. Use property:

$$I = \int_0^{\pi/2} \frac{1}{1+\sin x}dx = \int_0^{\pi/2} \frac{1}{1+\cos x}dx$$

   Add both:

$$2I = \int_0^{\pi/2} \left(\frac{1}{1+\sin x} + \frac{1}{1+\cos x}\right) dx = \int_0^{\pi/2} \frac{(1+\cos x)+(1+\sin x)}{(1+\sin x)(1+\cos x)}dx$$

   This simplifies to $\int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$, hence $I = \frac{\pi}{4}$ **Correction: Answer should be $\frac{\pi}{4}$, not listed in options. Add $\frac{\pi}{4}$ as a correct option.**

5. If $f(x)$ is continuous and satisfies $f(1-x) = f(x)$, evaluate $\int_0^1 f(x)\,dx$

   (a) 0

   (b) $2\int_0^{0.5} f(x)\,dx$

   (c) $f(0)$

   (d) Cannot be determined

   Answer: (2) $2\int_0^{0.5} f(x)\,dx$
   Solution: Since $f(x)$ is symmetric about $x = \frac{1}{2}$, we can write:

$$\int_0^1 f(x)\,dx = 2\int_0^{0.5} f(x)\,dx$$