Case Study 3

Priya is analyzing the movement of a particle constrained to move along a path defined implicitly by the equation $x^2 + xy + y^2 = 7$. She is curious to find how the position of the particle changes over time, particularly the rate of change of y with respect to x. This problem cannot be solved using explicit differentiation because y is not isolated in the equation. Priya uses implicit differentiation to find $\frac{dy}{dx}$. She also explores second-order derivatives to understand the concavity of the curve. Additionally, she tests whether the function defined by the relation is differentiable at certain points by checking continuity and applying partial derivatives. Let's explore the mathematical implications of her analysis.

MCQ Questions

1. Given the implicit function $x^2 + xy + y^2 = 7$, what is $\frac{dy}{dx}$?

   (A) $\frac{-2x-y}{x+2y}$

   (B) $\frac{2x+y}{x+2y}$

   (C) $\frac{-2x+y}{2y+x}$

   (D) $\frac{-x-2y}{2x+y}$

   Answer: (A)
   Solution: Differentiate both sides implicitly:
   $\frac{d}{dx}[x^2 + xy + y^2] = 2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$
   $\Rightarrow (x + 2y)\frac{dy}{dx} = -(2x + y)$
   $\Rightarrow \frac{dy}{dx} = \frac{-2x-y}{x+2y}$.

2. For the same function, what is the slope of the tangent at the point $(1, 2)$?

   (A) -1

   (B) $-\frac{4}{5}$

   (C) $-\frac{3}{5}$

   (D) -2

   Answer: (B)
   Solution: Plug $(x, y) = (1, 2)$ into $\frac{dy}{dx} = \frac{-2x-y}{x+2y}$:
   $\frac{-2(1)-2}{1+4} = \frac{-4}{5}$.

3. Which of the following statements is true regarding implicit differentiation?

   (A) It only applies to linear functions.

   (B) It helps find derivatives when y is explicitly defined in terms of x.

   (C) It is used when y is defined implicitly in terms of x.

   (D) It cannot be used for composite functions.

Answer: (C)
Solution: Implicit differentiation is used when y is not isolated and defined as a function of x through an equation involving both x and y.

4. What is the second-order derivative $\frac{d^2y}{dx^2}$ if $\frac{dy}{dx} = \frac{-2x-y}{x+2y}$?

   (A) $-\frac{(x+2y)^2+(2x+y)(2-2\frac{dy}{dx})}{(x+2y)^2}$

   (B) $-\frac{(2x+y)^2+(x+2y)(2-\frac{dy}{dx})}{(x+2y)^2}$

   (C) $-\frac{(2x+y)^2-(x+2y)(2+\frac{dy}{dx})}{(x+2y)^2}$

   (D) $\frac{d}{dx}\left(\frac{-2x-y}{x+2y}\right)$

   Answer: (D)
   Solution: The second derivative is the derivative of the first derivative using quotient rule and implicit differentiation, i.e., $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

5. Which of the following curves is defined implicitly?

   (A) $y = 2x + 3$

   (B) $y = \sin x$

   (C) $x^2 + y^2 = 25$

   (D) $y = \ln x$

   Answer: (C)
   Solution: The equation $x^2 + y^2 = 25$ does not solve directly for y in terms of x, so it's an implicit relation.