Case Study 5: Mastering Particular Integrals with Trigonometric Substitutions

Ananya, a dedicated student in Class 12, was revising particular integrals involving radical expressions and inverse trigonometric functions. She encountered integrals like $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ and $\int \frac{1}{a^2+x^2}\,dx$ and was initially confused. Her teacher explained that these have standard results and can also be solved using **trigonometric substitution**. For example, to evaluate $\int \frac{1}{\sqrt{4-x^2}}\,dx$, she was told to let $x = 2\sin\theta$ and use the identity $\sqrt{4-x^2} = 2\cos\theta$. She discovered how these integrals yield results like $\sin^{-1}\left(\frac{x}{a}\right)+C$ or $\tan^{-1}\left(\frac{x}{a}\right)+C$. This helped her tackle complex-looking expressions with confidence.

Key Standard Integrals:

- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
- Trigonometric substitutions:
  - $x = a\sin\theta$
  - $x = a\tan\theta$
  - $x = a\sec\theta$

MCQ Questions:

1. Evaluate $\int \frac{1}{\sqrt{9-x^2}}\,dx$.

   (a) $\sin^{-1}\left(\frac{x}{3}\right) + C$
   (b) $\cos^{-1}\left(\frac{x}{3}\right) + C$
   (c) $\ln|x + \sqrt{9-x^2}| + C$
   (d) $\tan^{-1}\left(\frac{x}{3}\right) + C$

   Answer: (a)
   Solution: It matches the form $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$. Here, $a = 3$, so the result is $\sin^{-1}\left(\frac{x}{3}\right) + C$

2. What is $\int \frac{1}{x^2+25}\,dx$?

   (a) $\tan^{-1}(x) + C$
   (b) $\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C$
   (c) $\ln|x^2+25| + C$
   (d) $\tan^{-1}\left(\frac{x}{25}\right) + C$

   Answer: (b)
   Solution: Using the standard result:

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C, \quad a = 5$$

3. Which substitution is suitable for $\int \frac{1}{\sqrt{x^2+4}}\,dx$?

(a) $x = 2\sin\theta$

(b) $x = 2\tan\theta$

(c) $x = 2\sec\theta$

(d) $x = 2\cos\theta$

Answer: (b)
Solution: When dealing with $\sqrt{x^2 + a^2}$, the substitution $x = a\tan\theta$ is used.

4. What is $\int \frac{1}{\sqrt{x^2-16}}\,dx$?

(a) $\sin^{-1}\left(\frac{x}{4}\right) + C$

(b) $\ln\left|x + \sqrt{x^2 - 16}\right| + C$

(c) $\tan^{-1}\left(\frac{x}{4}\right) + C$

(d) $\frac{1}{4}\ln\left|x + \sqrt{x^2 - 16}\right| + C$

Answer: (b)
Solution: This matches the form $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$, where $a = 4$. So the result is $\ln\left|x + \sqrt{x^2 - 16}\right| + C$

5. Evaluate $\int \frac{dx}{4+x^2}$.

(a) $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

(b) $\tan^{-1}\left(\frac{x}{4}\right) + C$

(c) $\ln|x^2 + 4| + C$

(d) $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$

Answer: (a)
Solution: Write $4 + x^2 = x^2 + 2^2$. So,

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Here, $a = 2 \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$