Case Study 3

In three-dimensional space, the orientation of a line is an important concept in understanding the geometry of objects and their relationships. One way to describe this orientation is through direction cosines and direction ratios. Direction cosines are the cosines of angles that a line makes with the positive directions of the coordinate axes. If a line makes angles α, β, and γ with the x, y, and z-axes respectively, then the direction cosines are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ which satisfy the identity:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Direction ratios, on the other hand, are any set of numbers proportional to the direction cosines. They are useful in writing the vector or Cartesian equations of a line. If a line passes through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then the direction ratios of the line are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. These can then be normalized to obtain the direction cosines. Understanding this is fundamental in 3D geometry and critical in applications involving angles, projections, and orientation in space.

1. If a line makes angles α, β, and γ with the coordinate axes, then which of the following is true?

   (a) $\cos\alpha + \cos\beta + \cos\gamma = 1$

   (b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

   (c) $\cos\alpha\cos\beta\cos\gamma = 1$

   (d) $\cos^2\alpha + \cos\beta + \cos^2\gamma = 1$

   Answer: (b)

   Solution: By definition of direction cosines, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

2. The direction ratios of a line are $3, 4, 12$. What are the direction cosines?

   (a) $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$

   (b) $\left(\frac{3}{14}, \frac{4}{14}, \frac{12}{14}\right)$

   (c) $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$

   (d) $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$

   Answer: (b)

   Solution: Magnitude $= \sqrt{3^2 + 4^2 + 12^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$.

   Direction cosines $= \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$.

   Option (b) is the correct choice.

3. The direction cosines of a line are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Find the angle that this line makes with the z-axis.

   (a) $30°$

   (b) $45°$

   (c) $60°$

   (d) $90°$

Answer: (b)

Solution: $\cos\gamma = \frac{1}{\sqrt{3}} \Rightarrow \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74°$.

None of the options match exactly; the correct answer should be approximately 54.74°.

Correction: Modify option to include approximately 54.74°.

4. If a line passes through points $A(1,2,3)$ and $B(4,6,9)$, then the direction ratios of the line are:

   (a) $(3,4,6)$
   (b) $(4,6,9)$
   (c) $(1,2,3)$
   (d) $(2,4,6)$

   Answer: (a)

   Solution: Direction ratios $= (4-1, 6-2, 9-3) = (3,4,6)$.

5. Which of the following cannot be direction cosines of a line?

   (a) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
   (b) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$
   (c) $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$
   (d) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

   Answer: (c)

   Solution: Check if sum of squares equals 1:

$$\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = \frac{9}{9} = 1$$

   So option (c) is valid. All options are valid direction cosines.

   Correction: Modify question to have one invalid option, e.g., (3, 4, 5).

   Replace (c) with: $\left(\frac{3}{5}, \frac{4}{5}, \frac{5}{5}\right) \Rightarrow \frac{9+16+25}{25} = \frac{50}{25} = 2 \neq 1 \rightarrow$ Invalid.

   Revised Answer: (c)