Case Study 5: Cross Product and Its Applications

In a robotics competition, a team builds a robotic arm that moves in three-dimensional space. To program rotations and torque, the students need to understand the vector cross product. They study how the cross product gives a vector perpendicular to two given vectors and how its magnitude can represent the area of a parallelogram or triangle. Using the right-hand rule, they determine the direction of rotation. They also analyze the torque $\vec{\tau} = \vec{r} \times \vec{F}$ and how vector product plays a role in rotational physics. Understanding cross product allows them to model forces and movements accurately in 3D space.

Theory and Formulae Related to Cross Product:

- Cross product: $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$
- Determinant form:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

- Right-hand rule gives direction of $\vec{a} \times \vec{b}$
- Area of parallelogram: $|\vec{a} \times \vec{b}|$
- Area of triangle: $\frac{1}{2}|\vec{a} \times \vec{b}|$
- If $\vec{a} \times \vec{b} = \vec{0}$, then vectors are parallel or collinear

MCQ Questions

1. If $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 3\hat{i} + \hat{j}$, find $\vec{a} \times \vec{b}$

   (a) $5\hat{k}$

   (b) $-5\hat{k}$

   (c) $4\hat{k}$

   (d) $-4\hat{k}$

   Answer: (a)
   Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix} = (2 \cdot 0 - 0 \cdot 1)\hat{i} - (1 \cdot 0 - 0 \cdot 3)\hat{j} + (1 \cdot 1 - 2 \cdot 3)\hat{k} = -5\hat{k}$$

   Correction: **Answer is (b)**.

2. What is the angle between two non-zero vectors if their cross product is zero?

(a) 30°

(b) 90°

(c) 180° or 0°

(d) Cannot be determined

Answer: (c)
Solution: If $\vec{a} \times \vec{b} = \vec{0}$, then $\sin\theta = 0 \Rightarrow \theta = 0^\circ$ or 180°

3. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, find $\vec{a} \times \vec{b}$

(a) $2\hat{i} + 0\hat{j} - 2\hat{k}$

(b) $2\hat{i} + 2\hat{j}$

(c) $2\hat{j}$

(d) $2\hat{i} + 2\hat{k}$

Answer: (a)
Solution:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = (1\cdot1-1\cdot(-1))\hat{i}-(1\cdot1-1\cdot1)\hat{j}+(1\cdot(-1)-1\cdot1)\hat{k} = 2\hat{i}+0\hat{j}-2\hat{k}$$

4. What is the area of a triangle formed by vectors $\vec{a} = 2\hat{i} + 3\hat{j}$ and $\vec{b} = \hat{i} + 4\hat{j}$?

(a) $\frac{1}{2}$

(b) $\frac{7}{2}$

(c) $\frac{5}{2}$

(d) $\frac{9}{2}$

Answer: (b)
Solution:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ 1 & 4 & 0 \end{vmatrix} = (3\cdot0-0\cdot4)\hat{i}-(2\cdot0-0\cdot1)\hat{j}+(2\cdot4-3\cdot1)\hat{k} = \hat{k}(8-3) = 5\hat{k} \Rightarrow \text{Area} = \frac{1}{2}\cdot|5| = \frac{5}{2}$$

Correction: **Answer is (c)**.

5. The cross product of two vectors is a:

(a) Scalar quantity

(b) Vector quantity

(c) Unitless number

(d) Zero always

Answer: (b)
Solution: Cross product yields a vector perpendicular to both original vectors