Case Study 2: Aircraft Navigation System

An aircraft navigation system is being designed to ensure safe flight paths over a mountainous region. A radar station is located at point $P(1, 2, 3)$, and a signal beacon is at $Q(4, 6, 7)$. The aircraft's flight path is along a line passing through P with direction ratios $(1, -2, 2)$. A restricted airspace plane is defined by $3x - y + 2z = 5$. Engineers need to determine the direction cosines of the flight path, the equation of the path, the angle between the path and the line PQ, the distance of the beacon from the plane, and conditions for coplanarity with another flight path to avoid collisions.

Key Formulas and Properties

- **Direction Cosines:** For a vector $a\hat{i}+b\hat{j}+c\hat{k}$, direction cosines are $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
- **Equation of a Line:** Vector form: $\vec{r} = \vec{a} + \lambda\vec{b}$; Cartesian form: $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.
- **Angle between Two Lines:** For direction vectors (a_1, b_1, c_1) and (a_2, b_2, c_2), $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$.
- **Distance from a Point to a Plane:** For point (x_1, y_1, z_1) and plane $ax + by + cz + d = 0$, distance $= \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$.
- **Coplanarity Condition:** Two lines with direction vectors $\vec{b_1}, \vec{b_2}$ and a vector joining points on them $\vec{a_2} - \vec{a_1}$ are coplanar if $[\vec{a_2} - \vec{a_1}, \vec{b_1}, \vec{b_2}] = 0$.

MCQ Questions

1. What are the direction cosines of the flight path with direction ratios $(1, -2, 2)$?

   a) $\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$
   b) $\left(\frac{1}{\sqrt{9}}, -\frac{2}{\sqrt{9}}, \frac{2}{\sqrt{9}}\right)$
   c) $\left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$
   d) $\left(\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

   Answer: a) $\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$
   Solution: The magnitude of the vector $(1, -2, 2)$ is $\sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$. Direction cosines are $\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}$. Verify: $\left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$. Option a is correct.

2. What is the Cartesian equation of the flight path passing through $P(1, 2, 3)$ with direction ratios $(1, -2, 2)$?

   a) $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{2}$
   b) $\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z-3}{1}$
   c) $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z-3}{2}$
   d) $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{-2}$

   Answer: a) $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{2}$
   Solution: The line passes through $(1, 2, 3)$ with direction ratios $(1, -2, 2)$. The Cartesian equation is $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{2}$. Option a matches this equation.

3. What is the distance from the beacon at $Q(4, 6, 7)$ to the plane $3x - y + 2z = 5$?

   a) $\frac{15}{\sqrt{14}}$

   b) $\frac{21}{\sqrt{14}}$

   c) $\frac{15}{\sqrt{13}}$

   d) $\frac{21}{\sqrt{13}}$

   Answer: a) $\frac{15}{\sqrt{14}}$

   Solution: For plane $3x - y + 2z - 5 = 0$, distance from $(4, 6, 7)$ is $\left|\frac{3\cdot4+(-1)\cdot6+2\cdot7-5}{\sqrt{3^2+(-1)^2+2^2}}\right| = \left|\frac{12-6+14-5}{\sqrt{9+1+4}}\right| = \left|\frac{15}{\sqrt{14}}\right| = \frac{15}{\sqrt{14}}$. Option a is correct.

4. What is the angle between the flight path and the line joining $P(1, 2, 3)$ to $Q(4, 6, 7)$?

   a) $\cos^{-1}\left(\frac{3}{\sqrt{9}\cdot\sqrt{41}}\right)$

   b) $\cos^{-1}\left(\frac{9}{\sqrt{9}\cdot\sqrt{41}}\right)$

   c) $\cos^{-1}\left(\frac{3}{\sqrt{41}\cdot\sqrt{9}}\right)$

   d) $\cos^{-1}\left(\frac{9}{\sqrt{41}\cdot\sqrt{9}}\right)$

   Answer: b) $\cos^{-1}\left(\frac{9}{\sqrt{9}\cdot\sqrt{41}}\right)$

   Solution: Direction vector of line PQ: $(4-1, 6-2, 7-3) = (3, 4, 4)$. Flight path direction vector: $(1, -2, 2)$. Cosine of angle: $\cos\theta = \frac{1\cdot3+(-2)\cdot4+2\cdot4}{\sqrt{1^2+(-2)^2+2^2}\cdot\sqrt{3^2+4^2+4^2}} = \frac{3-8+8}{\sqrt{9}\cdot\sqrt{41}} = \frac{3}{3\cdot\sqrt{41}} = \frac{1}{\sqrt{41}}$. This matches option b: $\frac{3}{\sqrt{9}\cdot\sqrt{41}} = \frac{1}{\sqrt{41}}$.

5. What is the condition for the flight path to be coplanar with another line passing through $Q(4, 6, 7)$ with direction ratios $(2, 1, -1)$?

   a) $\begin{vmatrix} 3 & 4 & 4 \\ 1 & -2 & 2 \\ 2 & 1 & -1 \end{vmatrix} = 0$

   b) $\begin{vmatrix} 1 & -2 & 2 \\ 3 & 4 & 4 \\ 2 & 1 & -1 \end{vmatrix} = 0$

   c) $\begin{vmatrix} 4 & 6 & 7 \\ 1 & -2 & 2 \\ 2 & 1 & -1 \end{vmatrix} = 0$

   d) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 2 & 1 & -1 \end{vmatrix} = 0$

   Answer: a) $\begin{vmatrix} 3 & 4 & 4 \\ 1 & -2 & 2 \\ 2 & 1 & -1 \end{vmatrix} = 0$

   Solution: For coplanarity, the scalar triple product of vectors $\vec{QP} = (3, 4, 4)$, direction vector $(1, -2, 2)$, and direction vector $(2, 1, -1)$ must be zero. The determinant is $\begin{vmatrix} 3 & 4 & 4 \\ 1 & -2 & 2 \\ 2 & 1 & -1 \end{vmatrix} = 3((-2)\cdot(-1) - 2\cdot1) - 4(1\cdot(-1) - 2\cdot2) + 4(1\cdot1 - (-2)\cdot2) =$

$3(2-2)-4(-1-4)+4(1+4)=0+20+20=40\neq 0$. Option a is the correct form, though the lines are not coplanar.