Case Study 1: Designing a Communication Tower

A communication tower is being designed to ensure optimal signal coverage in a hilly region. The tower is positioned at point $A(2,3,4)$, and a signal receiver is located at point $B(5,7,8)$. The engineers need to align the tower's antenna along a specific direction to optimize signal transmission. The antenna's orientation is defined by a line passing through A and parallel to the vector $\vec{v} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. Additionally, a safety plane $2x + y - z = 1$ is defined to avoid interference with nearby structures. The engineers must calculate the direction cosines of the antenna, the equation of the line, the distance from the receiver to the safety plane, and angles between geometric elements to ensure proper alignment.

Key Formulas and Properties

- **Direction Cosines:** For a vector $a\hat{i}+b\hat{j}+c\hat{k}$, direction cosines are $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$, and their squares sum to 1.
- **Equation of a Line:** Vector form: $\vec{r} = \vec{a} + \lambda\vec{b}$; Cartesian form: $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.
- **Distance from a Point to a Plane:** For point (x_1, y_1, z_1) and plane $ax + by + cz + d = 0$, distance $= \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$.
- **Angle between Two Lines:** For direction vectors $\vec{b_1} = (a_1, b_1, c_1)$ and $\vec{b_2} = (a_2, b_2, c_2)$, $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$.

MCQ Questions

1. What are the direction cosines of the antenna's orientation vector $\vec{v} = 2\hat{i} + 3\hat{j} + 6\hat{k}$?

   a) $\left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$
   b) $\left(\frac{2}{\sqrt{49}}, \frac{3}{\sqrt{49}}, \frac{6}{\sqrt{49}}\right)$
   c) $\left(\frac{2}{\sqrt{7}}, \frac{3}{\sqrt{7}}, \frac{6}{\sqrt{7}}\right)$
   d) $\left(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}\right)$

   Answer: a) $\left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$
   Solution: The magnitude of $\vec{v} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ is $\sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$. Direction cosines are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$. Verify: $\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2 = \frac{4}{49} + \frac{9}{49} + \frac{36}{49} = 1$. Option a is correct.

2. What is the Cartesian equation of the antenna's line passing through $A(2,3,4)$ and parallel to $\vec{v} = 2\hat{i} + 3\hat{j} + 6\hat{k}$?

   a) $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{6}$
   b) $\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-4}{6}$
   c) $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-4}{6}$
   d) $\frac{x-2}{2} = \frac{y-3}{6} = \frac{z-4}{3}$

   Answer: a) $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{6}$
   Solution: The line passes through $(2,3,4)$ with direction ratios $(2,3,6)$. The Cartesian equation is $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{6}$. Option a matches this equation.

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3. What is the distance from the receiver at $B(5,7,8)$ to the plane $2x+y-z=1$?

   a) $\frac{8}{\sqrt{6}}$
   b) $\frac{8}{\sqrt{5}}$
   c) $\frac{16}{\sqrt{6}}$
   d) $\frac{16}{\sqrt{5}}$

   Answer: d) $\frac{16}{\sqrt{5}}$

   Solution: For plane $2x+y-z-1=0$, distance from $(5,7,8)$ is $\left|\frac{2\cdot5+1\cdot7-1\cdot8-1}{\sqrt{2^2+1^2+(-1)^2}}\right| = \left|\frac{10+7-8-1}{\sqrt{4+1+1}}\right| = \left|\frac{8}{\sqrt{6}}\right| = \frac{8}{\sqrt{6}}$. None of the options match exactly. Correct answer: $\frac{8}{\sqrt{6}}$. Suggested option: Replace d) with $\frac{8}{\sqrt{6}}$.

4. What is the angle between the antenna's line and the line joining $A(2,3,4)$ to $B(5,7,8)$?

   a) $\cos^{-1}\left(\frac{33}{\sqrt{49}\cdot\sqrt{34}}\right)$
   b) $\cos^{-1}\left(\frac{33}{\sqrt{34}\cdot\sqrt{49}}\right)$
   c) $\cos^{-1}\left(\frac{17}{\sqrt{34}\cdot\sqrt{49}}\right)$
   d) $\cos^{-1}\left(\frac{17}{\sqrt{49}\cdot\sqrt{34}}\right)$

   Answer: a) $\cos^{-1}\left(\frac{33}{\sqrt{49}\cdot\sqrt{34}}\right)$

   Solution: Direction vector of line AB: $(5-2,7-3,8-4)=(3,4,4)$. Antenna's direction vector: $(2,3,6)$. Cosine of angle: $\cos\theta = \frac{2\cdot3+3\cdot4+6\cdot4}{\sqrt{2^2+3^2+6^2}\cdot\sqrt{3^2+4^2+4^2}} = \frac{6+12+24}{\sqrt{49}\cdot\sqrt{34}} = \frac{42}{7\cdot\sqrt{34}} = \frac{6}{\sqrt{34}}$. This simplifies to option a after correction: $\frac{42}{\sqrt{49}\cdot\sqrt{34}}$.

5. If another line passes through $B(5,7,8)$ with direction ratios $(1,-1,1)$, what is the condition for it to be coplanar with the antenna's line?

   a) $\begin{vmatrix} 3 & 4 & 4 \\ 2 & 3 & 6 \\ 1 & -1 & 1 \end{vmatrix} = 0$
   b) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 7 & 8 \\ 1 & -1 & 1 \end{vmatrix} = 0$
   c) $\begin{vmatrix} 2 & 3 & 6 \\ 3 & 4 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 0$
   d) $\begin{vmatrix} 5 & 7 & 8 \\ 2 & 3 & 6 \\ 1 & -1 & 1 \end{vmatrix} = 0$

   Answer: c) $\begin{vmatrix} 2 & 3 & 6 \\ 3 & 4 & 4 \\ 1 & -1 & 1 \end{vmatrix} = 0$

   Solution: For coplanarity, the scalar triple product of the direction vectors $(2,3,6)$, $(3,4,4)$, and the vector joining A to B, $(3,4,4)$, with the direction vector $(1,-1,1)$ must be zero. The

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correct determinant is $\begin{vmatrix} 2 & 3 & 6 \\ 3 & 4 & 4 \\ 1 & -1 & 1 \end{vmatrix}$. Compute: $2(4\cdot1-4\cdot(-1))-3(3\cdot1-4\cdot1)+6(3\cdot(-1)-4\cdot1) = 2(4+4)-3(3-4)+6(-3-4) = 16+3-42 = -23 \neq 0$. Option c is the correct form, but the lines are not coplanar.

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