Case Study 2: Medical Diagnosis Reliability (Probability)

Case Study Description:

A public health department is evaluating a new, non-invasive diagnostic test for a very rare genetic disorder, let's call it Disease D. The prevalence of Disease D in the general population is estimated to be 0.5% (or one in two hundred people). The test is known to be highly accurate. Specifically, the probability that the test correctly identifies a person **with** the disease (a true positive result) is 99%. However, no test is perfect. The probability that the test correctly identifies a person **without** the disease (a true negative result) is 98%. This means there is a chance of a false positive (testing positive when healthy) and a false negative (testing negative when sick). A person is selected at random from the population and is administered this test. The public health officials are particularly interested in the probability that a person who tests positive actually has the disease, as this impacts resource allocation and patient counseling. This scenario highlights how **Conditional Probability** and the concept of **Bayes' Theorem** are crucial for interpreting medical test results, especially for rare diseases, where the prior probability is very low. Even a highly accurate test can yield surprising results when applied to the general population.

Let D be the event that a person has the disease, and T be the event that the test result is positive. We are given $P(D) = 0.005$, $P(T|D) = 0.99$, and $P(T'|D') = 0.98$.

MCQ Questions (5 Questions)

1. What is the probability of a **false positive** result, which is the probability that a person tests positive given that they do not have the disease?

   (a) 0.01
   (b) 0.99
   (c) 0.02
   (d) 0.98

   Answer: (c)

   Solution: Let D' be the event the person does not have the disease. Let T be the event of testing positive. A false positive is the event T given D', $P(T|D')$. We are given the probability of a true negative, $P(T'|D') = 0.98$. Since T and T' are complementary events, the conditional probabilities must sum to 1:

$$P(T|D') = 1 - P(T'|D') = 1 - 0.98 = 0.02$$

2. What is the probability that a randomly selected person is **disease-free and tests positive**? (Multiplication Theorem)

   (a) 0.00010
   (b) 0.00495
   (c) 0.01990
   (d) 0.97515

   Answer: (c)

   Solution: We need to find $P(D' \cap T)$. This is the probability of a false positive case. We use the **Multiplication Theorem of Probability**:

$$P(D' \cap T) = P(D') \cdot P(T|D')$$

We have $P(D) = 0.005$, so $P(D') = 1 - 0.005 = 0.995$. From Q1, $P(T|D') = 0.02$.

$$P(D' \cap T) = 0.995 \times 0.02 = 0.01990$$

3. What is the probability that a randomly selected person from the population tests positive for the disease? (Total Probability)

   (a) 0.020

   (b) 0.02485

   (c) 0.01990

   (d) 0.005

   Answer: (b)

   Solution: The total probability of the test being positive, $P(T)$, is found using the **Theorem of Total Probability**:

   $$P(T) = P(D)P(T|D) + P(D')P(T|D')$$

   $P(D)P(T|D) = 0.005 \times 0.99 = 0.00495$ (True Positive probability) $P(D')P(T|D') = 0.01990$ (from Q2)

   $$P(T) = 0.00495 + 0.01990 = 0.02485$$

4. If a person tests positive, what is the probability that they **actually have the disease**? (Bayes' Theorem)

   (a) $\frac{99}{497}$

   (b) $\frac{398}{497}$

   (c) $\frac{5}{1000}$

   (d) $\frac{199}{2485}$

   Answer: (a)

   Solution: We need to find the conditional probability $P(D|T)$, which is calculated using **Bayes' Theorem**:

   $$P(D|T) = \frac{P(D \cap T)}{P(T)} = \frac{P(D) \cdot P(T|D)}{P(T)}$$

   From Q3, we have: $P(D \cap T) = 0.00495$ $P(T) = 0.02485$

   $$P(D|T) = \frac{0.00495}{0.02485} = \frac{495}{2485}$$

   To simplify the fraction, divide the numerator and denominator by 5:

   $$\frac{495\boldsymbol{\nabla} \cdot 5}{2485\boldsymbol{\nabla} \cdot 5} = \frac{99}{497}$$

5. Suppose 10 people are randomly selected and tested. Let X be the random variable representing the number of people who test positive. What is the expected number (Mean) of people who will test positive? (Random Variable and Mean)

   (a) 10×0.005

(b) 10×0.995

(c) 10×0.02485

(d) 10×0.97515

Answer: (c)

Solution: The random variable X, the number of people who test positive out of $n = 10$ trials, follows a Binomial distribution $B(n, p)$, where $p = P(\text{Test Positive}) = P(T)$. From Q3, we have $P(T) = p = 0.02485$. The **Mean** (or Expected Value) of a Binomial distribution is given by $E(X) = n \cdot p$.

$$E(X) = 10 \times 0.02485$$

This calculation represents the average number of positive results expected in a sample of this size, based on the overall probability of a positive test.