Case Study 4: Applications of Differential Equations in Real Life

Differential equations are widely used to model phenomena such as population growth, radioactive decay, temperature change, and motion under gravity with resistance. The most common model for growth or decay is:

$$\frac{dy}{dt} = ky \Rightarrow y = Ae^{kt}$$

Where:

- y: quantity at time t
- A: initial quantity
- k: constant of proportionality (positive for growth, negative for decay)

In Newton's Law of Cooling, the rate of change of temperature is proportional to the difference between the object's temperature and ambient temperature:

$$\frac{dT}{dt} = -k(T - T_a)$$

Similarly, motion under gravity with resistance (proportional to velocity) follows:

$$m\frac{dv}{dt} = mg - kv$$

Understanding these models helps solve practical problems in physics, engineering, and biology.

MCQ Questions

1. The solution of the differential equation $\frac{dy}{dt} = ky$, $y(0) = A$, is:

   (a) $y = A + kt$

   (b) $y = Ae^{kt}$

   (c) $y = Ae^{-kt}$

   (d) $y = Ae^{k/t}$

   Answer: (b) $y = Ae^{kt}$

   Solution: This is a standard exponential growth/decay model. The general solution is $y = Ae^{kt}$, where A is the initial value.

2. A body cools from $80°C$ to $60°C$ in 5 minutes. If ambient temperature is $20°C$, which model applies?

   (a) $\frac{dT}{dt} = kT$

   (b) $\frac{dT}{dt} = -kT$

   (c) $\frac{dT}{dt} = -k(T - 20)$

(d) $\frac{dT}{dt} = k(T - 20)$

Answer: (c) $\frac{dT}{dt} = -k(T - 20)$

Solution: According to Newton's Law of Cooling, the rate of cooling is proportional to the temperature difference from ambient temperature.

3. In a decaying process, the population reduces by half every hour. If initial population is 1000, what is the population after 3 hours?

(a) 500

(b) 250

(c) 125

(d) 100

Answer: (c) 125

Solution: Exponential decay with half-life of 1 hour:

$$y = 1000 \cdot \left(\frac{1}{2}\right)^3 = 125$$

4. In motion under gravity with resistance, the velocity satisfies:

$$m\frac{dv}{dt} = mg - kv$$

The terminal velocity is:

(a) $v = \frac{mg}{k}$

(b) $v = \frac{k}{mg}$

(c) $v = mgk$

(d) $v = \frac{g}{k}$

Answer: (a) $v = \frac{mg}{k}$

Solution: At terminal velocity $dv/dt = 0 \Rightarrow mg = kv \Rightarrow v = \frac{mg}{k}$

5. A radioactive substance decays with $k = -0.01$. If $A = 500$, what is the quantity after 100 units of time?

(a) 184.0

(b) 135.3

(c) 200.5

(d) 300.1

Answer: (b) 135.3

Solution:

$$y = 500e^{-0.01\cdot 100} = 500e^{-1} \approx 500 \cdot 0.3679 \approx 183.95$$

Correction: The answer should be (a) 184.0 (rounded), not (b). So,

Correct Answer: (a) 184.0