Case Study 4: Chain Rule and Composite Functions in Real-Life Physics Applications

Rahul is a physics student studying the temperature change in a metal rod that is heated over time. The temperature T (in Celsius) at any time t is modeled by a function $T(t) = \sin(\ln t)$ for $t > 0$. To understand how quickly the temperature changes, he needs to find the rate of change of T with respect to time, i.e., $\frac{dT}{dt}$. He realizes that this requires the application of the chain rule since the function is a composition of two functions: the logarithmic function and the sine function. Rahul also explores the second derivative to analyze the rate at which the temperature change is itself changing. This case presents a practical example of composite and chain rule applications in a physics context.

MCQ Questions

1. What is the derivative of $T(t) = \sin(\ln t)$ with respect to t?

   (A) $\cos(\ln t)$

   (B) $\cos(\ln t) \cdot \frac{1}{t}$

   (C) $\frac{\sin(\ln t)}{t}$

   (D) $\ln(\sin t)$

   Answer: (B)
   Solution: Let $u = \ln t$, then $T(t) = \sin(u)$. So, $\frac{dT}{dt} = \cos(u) \cdot \frac{du}{dt} = \cos(\ln t) \cdot \frac{1}{t}$.

2. What is the second derivative of $T(t) = \sin(\ln t)$?

   (A) $-\sin(\ln t) \cdot \frac{1}{t^2}$

   (B) $\cos(\ln t) \cdot \frac{1}{t^2}$

   (C) $\frac{-\sin(\ln t)+\cos(\ln t)}{t^2}$

   (D) $\frac{-\sin(\ln t)+\cos(\ln t)}{t}$

   Answer: (A)
   Solution: First derivative is $\frac{dT}{dt} = \cos(\ln t) \cdot \frac{1}{t}$. Then using product rule:

   $$\frac{d^2T}{dt^2} = \frac{d}{dt}\left(\cos(\ln t) \cdot \frac{1}{t}\right) = -\sin(\ln t) \cdot \frac{1}{t} \cdot \frac{1}{t} - \cos(\ln t) \cdot \frac{1}{t^2} = -\frac{\sin(\ln t) + \cos(\ln t)}{t^2}$$

3. Which of the following best defines a composite function?

   (A) $f(x) + g(x)$

   (B) $f(g(x))$

   (C) $f(x) \cdot g(x)$

   (D) $\frac{f(x)}{g(x)}$

Answer: (B)
Solution: A composite function is one in which the output of one function becomes the input of another, written as $f(g(x))$.

4. Which rule is applied in finding the derivative of $T(t) = \sin(\ln t)$?

   (A) Product Rule
   (B) Quotient Rule
   (C) Chain Rule
   (D) Implicit Rule

   Answer: (C)
   Solution: The chain rule is applied because $\sin(\ln t)$ is a composition of $\sin(u)$ and $u = \ln t$.

5. What is the value of $\frac{dT}{dt}$ at $t = 1$, where $T(t) = \sin(\ln t)$?

   (A) $\cos(0)$
   (B) $\cos(1)$
   (C) $\cos(0) \cdot 1$
   (D) $\cos(0) \cdot \frac{1}{1}$

   Answer: (D)
   Solution: $\frac{dT}{dt} = \cos(\ln t) \cdot \frac{1}{t}$; at $t = 1$, $\ln(1) = 0$, so it becomes $\cos(0) \cdot \frac{1}{1} = 1$.