Case Study 5: Differentiability and Continuity of an Inverse Trigonometric Function

Sanya is exploring the properties of the function $f(x) = \arctan\left(\frac{2x}{1-x^2}\right)$, which is an example of an inverse trigonometric function composed with a rational function. She wants to analyze whether this function is continuous and differentiable over the domain where it is defined. She also aims to find the derivative of $f(x)$ using implicit differentiation and apply the chain rule to verify her results. Sanya investigates the behavior of the function at critical points, including where the denominator in the argument of arctan becomes zero. This helps her understand the function's smoothness and the nature of its graph.

MCQ Questions

1. What is the derivative of $f(x) = \arctan\left(\frac{2x}{1-x^2}\right)$?

   (A) $\frac{2(1+x^2)}{(1-x^2)^2+4x^2}$

   (B) $\frac{2(1-x^2)}{(1+x^2)^2+4x^2}$

   (C) $\frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2} \cdot \frac{2(1+x^2)}{(1-x^2)^2}$

   (D) $\frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2} \cdot \frac{2(1-x^2)}{(1+x^2)^2}$

   Answer: (C)
   Solution: Using chain rule:

   $$f'(x) = \frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2} \cdot \frac{d}{dx}\left(\frac{2x}{1-x^2}\right).$$

   Differentiate the inner function using quotient rule:

   $$\frac{d}{dx}\left(\frac{2x}{1-x^2}\right) = \frac{2(1-x^2)+2x(2x)}{(1-x^2)^2} = \frac{2(1+x^2)}{(1-x^2)^2}.$$

2. What is the domain of $f(x) = \arctan\left(\frac{2x}{1-x^2}\right)$?

   (A) All real x except $x = \pm 1$

   (B) All real x

   (C) Only $x > 0$

   (D) Only $x < 1$

   Answer: (A)
   Solution: The function is undefined where the denominator of the argument is zero, i.e., $1 - x^2 = 0 \Rightarrow x = \pm 1$.

3. Is the function $f(x) = \arctan\left(\frac{2x}{1-x^2}\right)$ continuous at $x = 0$?

(A) Yes, it is continuous and differentiable at $x = 0$

(B) No, it is discontinuous at $x = 0$

(C) Continuous but not differentiable at $x = 0$

(D) Neither continuous nor differentiable at $x = 0$

Answer: (A)
Solution: The function is defined and smooth at $x = 0$; hence continuous and differentiable there.

4. What is the value of $f(0)$?

(A) 0

(B) $\frac{\pi}{2}$

(C) 1

(D) Undefined

Answer: (A)
Solution: $f(0) = \arctan\left(\frac{0}{1-0}\right) = \arctan(0) = 0.$

5. Which rule is primarily used to differentiate inverse trigonometric functions like $f(x)$?

(A) Chain Rule

(B) Product Rule

(C) Quotient Rule only

(D) Integration by Parts

Answer: (A)
Solution: Differentiation of composite inverse trig functions requires the chain rule combined with quotient rule for inner functions.