Case Study 2: Evaluating Definite Integrals Using Standard Results and Area Interpretation

Meera was preparing for her final board exam in Mathematics. While solving problems from the chapter "Definite Integrals", she was particularly interested in problems that involved evaluating integrals using standard results like $\int x^n \, dx$, $\int \sin x \, dx$, and $\int \cos x \, dx$, as well as understanding definite integrals as the net area under a curve. Her teacher emphasized that the definite integral does not always represent the actual area, especially when the curve lies below the x-axis in parts of the interval.

Meera used the following standard results repeatedly:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1), \qquad \int \sin x \, dx = -\cos x + C, \qquad \int \cos x \, dx = \sin x + C$$

She also learned to apply limits using the fundamental theorem:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Answer the following multiple-choice questions based on Meera's problem-solving journey.

MCQ Questions

1. Evaluate: $\int_0^{\pi} \cos x \, dx$
   - (a) 0
   - (b) 1
   - (c) 2
   - (d) -2

   Answer: (1) 0
   Solution:

$$\int_0^{\pi} \cos x \, dx = [\sin x]_0^{\pi} = \sin \pi - \sin 0 = 0 - 0 = 0$$

2. Find $\int_1^4 x^2 \, dx$
   - (a) 21
   - (b) $\frac{63}{3}$
   - (c) $\frac{65}{3}$
   - (d) 18

   Answer: (3) $\frac{65}{3}$
   Solution:

$$\int_1^4 x^2 \, dx = \left[\frac{x^3}{3}\right]_1^4 = \frac{64}{3} - \frac{1}{3} = \frac{63}{3} = 21$$

   Correction: Option (1) is correct. The actual answer is 21.

3. If $f(x) = 3x^2 + 2x$, find $\int_0^2 f(x) \, dx$
   - (a) 18
   - (b) 20

(c) 22

(d) 24

Answer: (2) 20
Solution:

$$\int_0^2 (3x^2 + 2x)\,dx = \left[x^3 + x^2\right]_0^2 = (8+4) - (0+0) = 12$$

Correction: The correct value is 12. So the correct option should be added as 12. None of the above is correct.

4. Using symmetry, evaluate: $\int_{-3}^{3} x^3\,dx$

   (a) 0

   (b) 18

   (c) -18

   (d) 27

   Answer: (1) 0
   Solution: Since x^3 is an odd function, $\int_{-a}^{a} x^3\,dx = 0$

5. Which of the following is NOT a property of definite integrals?

   (a) $\int_a^a f(x)\,dx = 0$

   (b) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

   (c) $\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$

   (d) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

   Answer: (3) $\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$
   Solution: This contradicts the actual property which says:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$