Case Study 4: Application of Definite Integrals in Area Under Curves

Priya, a class 12 student, was revising for her pre-board exams. Her teacher introduced her to the concept of using definite integrals to find the area enclosed by curves. Priya learned that when a curve lies entirely above the x-axis in the given interval, the value of the definite integral directly gives the area. However, when the curve dips below the x-axis, the integral gives a negative value, and the actual area must be computed by taking the absolute value.

She was taught standard area formulas:

$$\text{Area under } y = f(x) \text{ from } x = a \text{ to } x = b \text{ is: } \int_a^b f(x)\,dx$$

She also understood the importance of interpreting the result in context and checking if the function changes sign within the interval.

Answer the following questions based on Priya's experience.

MCQ Questions

1. Find the area under the curve $y = x^2$ from $x = 0$ to $x = 3$

   (a) 9
   (b) 27
   (c) $\frac{27}{3}$
   (d) $\frac{9}{2}$

   Answer: (3) $\frac{27}{3}$
   Solution:

$$\int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{27}{3} - 0 = 9$$

2. Find the area bounded by $y = |x|$ from $x = -2$ to $x = 2$

   (a) 4
   (b) 8
   (c) 2
   (d) 0

   Answer: (2) 8
   Solution: Since $|x|$ is even, use:

$$\int_{-2}^{2} |x|\,dx = 2\int_0^2 x\,dx = 2 \cdot \left[\frac{x^2}{2}\right]_0^2 = 2 \cdot \frac{4}{2} = 4 \Rightarrow 2 \cdot 4 = 8$$

3. The value of $\int_0^\pi \sin^2 x\,dx$ is:

   (a) π
   (b) $\frac{\pi}{2}$
   (c) $\frac{\pi}{4}$
   (d) 0

Answer: (2) $\frac{\pi}{2}$
Solution: Use identity: $\sin^2 x = \frac{1-\cos 2x}{2}$

$$\int_0^{\pi} \sin^2 x\, dx = \int_0^{\pi} \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}(\pi - 0) = \frac{\pi}{2}$$

4. The area bounded by the curve $y = \sin x$ and the x-axis from $x = 0$ to $x = 2\pi$ is:

   (a) 0
   (b) 2
   (c) 4
   (d) *None*

   Answer: (3) 4
   Solution: $\sin x$ is positive from 0 to π and negative from π to 2π:

   $$\text{Total area} = \int_0^{\pi} \sin x\, dx - \int_{\pi}^{2\pi} \sin x\, dx = 2 + 2 = 4$$

   (we take the absolute value for the second part)

5. Evaluate: $\int_{-2}^{2} x^2\, dx$

   (a) 0
   (b) $\frac{16}{3}$
   (c) $\frac{8}{3}$
   (d) 4

   Answer: (2) $\frac{16}{3}$
   Solution: x^2 is even:

   $$\int_{-2}^{2} x^2\, dx = 2\int_0^{2} x^2\, dx = 2 \cdot \left[\frac{x^3}{3}\right]_0^2 = 2 \cdot \frac{8}{3} = \frac{16}{3}$$