Case Study 1

Rahul, a student of Class 12, is interning at a robotics startup where he has been asked to study how the robotic arm's motion changes with temperature. The angle $\theta(t)$ (in radians) made by the robotic arm at any time t is a differentiable function of time, and temperature T affects the speed of its movement, i.e., θ also indirectly depends on temperature through time. The function θ is given by $\theta(t) = \ln(\cos t)$ and $T(t) = 2t + 5$. Rahul wants to study how the angular speed $\frac{d\theta}{dt}$ changes over time and how this relates to the continuity and differentiability of composite functions. Based on his findings, he notices that some values of t cause the function to become undefined. He decides to analyze the continuity and differentiability of such composite expressions, including identifying where the functions may break or behave unexpectedly due to domain restrictions. Let's explore some questions based on Rahul's observations.

MCQ Questions

1. For the function $\theta(t) = \ln(\cos t)$, which of the following time intervals ensures that $\theta(t)$ is defined and continuous?

   (A) $t \in \left(\frac{\pi}{2}, \pi\right)$

   (B) $t \in \left(0, \frac{\pi}{2}\right)$

   (C) $t \in \left(\pi, \frac{3\pi}{2}\right)$

   (D) $t \in \left(\frac{3\pi}{2}, 2\pi\right)$

   Answer: (B)
   Solution: $\cos t > 0$ is required for $\ln(\cos t)$ to be defined. This occurs in the interval $\left(0, \frac{\pi}{2}\right)$ within one full cycle of cosine function.

2. What is the derivative $\frac{d\theta}{dt}$ if $\theta(t) = \ln(\cos t)$?

   (A) $\frac{1}{\cos t}$

   (B) $-\tan t$

   (C) $\tan t$

   (D) $\sec t$

   Answer: (B)
   Solution: Using chain rule, $\frac{d}{dt}[\ln(\cos t)] = \frac{1}{\cos t} \cdot (-\sin t) = -\tan t$.

3. Which of the following functions is differentiable for all real numbers?

   (A) $f(x) = \ln(\cos x)$

   (B) $f(x) = \tan^{-1} x$

   (C) $f(x) = \sec x$

(D) $f(x) = \tan x$

Answer: (B)
Solution: $\tan^{-1} x$ is differentiable for all real x because it has no domain restrictions. The other functions are undefined at certain points.

4. Let $f(x) = \ln(\cos x)$ and $g(x) = 2x + 5$. What is $\frac{d}{dx}[f(g(x))]$?

   (A) $-2\tan(2x+5)$

   (B) $\sec(2x+5)$

   (C) $-\sin(2x+5)$

   (D) $\frac{-2\sin(2x+5)}{\cos(2x+5)}$

   Answer: (A)
   Solution: Using chain rule: $\frac{d}{dx}[\ln(\cos(2x+5))] = \frac{-\sin(2x+5)}{\cos(2x+5)} \cdot 2 = -2\tan(2x+5)$.

5. Which of the following statements is true regarding $f(x) = \ln(\cos x)$?

   (A) It is continuous for all real numbers.

   (B) It is discontinuous at $x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$.

   (C) It is not differentiable anywhere.

   (D) It is a polynomial.

   Answer: (B)
   Solution: $\ln(\cos x)$ is undefined (and hence discontinuous) wherever $\cos x \leq 0$, specifically at $x = \frac{\pi}{2} + n\pi$.