SOLUTIONS FOR SET 5

1. **Question:** If the equation $ax^2 + 2bx + c = 0$ and $x^2 + 2r^2x + 1 = 0$ have one common root and a, b, c are in A.P ($r^2 \neq 1$), then the roots of the equation $x^2 + 2r^2x + 1 = 0$ are

   Solution:

   (i) Since a, b, c are in A.P., we have $2b = a + c$.

   (ii) Let $f(x) = ax^2 + 2bx + c$ and $g(x) = x^2 + 2r^2x + 1$.

   (iii) If $f(x) = 0$ has a common root α with $g(x) = 0$, then α satisfies both equations:
   $$a\alpha^2 + 2b\alpha + c = 0 \quad (1)$$
   $$\alpha^2 + 2r^2\alpha + 1 = 0 \quad (2)$$

   (iv) Substitute $2b = a + c$ into (1):
   $$a\alpha^2 + (a + c)\alpha + c = 0$$
   $$a\alpha^2 + a\alpha + c\alpha + c = 0$$
   $$a\alpha(\alpha + 1) + c(\alpha + 1) = 0$$
   $$(\alpha + 1)(a\alpha + c) = 0$$

   (v) The roots of $f(x) = 0$ are $\alpha = -\mathbf{1}$ and $\alpha = -\mathbf{c}/\mathbf{a}$.

   (vi) Since $f(x) = 0$ and $g(x) = 0$ have one common root, this common root must be either -1 or $-c/a$.

   (vii) **Case A: $\alpha = -1$ is the common root.** Substitute $\alpha = -1$ into $g(x) = 0$:
   $$(-1)^2 + 2r^2(-1) + 1 = 0$$
   $$1 - 2r^2 + 1 = 0 \implies 2r^2 = 2 \implies r^2 = 1$$
   This contradicts the given condition $r^2 \neq 1$. Hence, $x = -1$ cannot be the common root.

   (viii) **Case B: $\alpha = -c/a$ is the common root.** Substitute $\alpha = -c/a$ into $g(x) = 0$:
   $$\left(-\frac{c}{a}\right)^2 + 2r^2\left(-\frac{c}{a}\right) + 1 = 0$$
   $$\frac{c^2}{a^2} - \frac{2r^2c}{a} + 1 = 0$$
   Multiply by a^2:
   $$c^2 - 2r^2ac + a^2 = 0 \quad (3)$$

   (ix) The roots of $g(x) = x^2 + 2r^2x + 1 = 0$ are given by the quadratic formula:
   $$x = \frac{-2r^2 \pm \sqrt{(2r^2)^2 - 4(1)(1)}}{2} = -r^2 \pm \sqrt{r^4 - 1}$$
   Let the roots of $g(x) = 0$ be α_1, α_2. Their product is $\alpha_1\alpha_2 = 1$. The common root is $\alpha = -c/a$. Since $g(x) = 0$ has two roots, $x = -c/a$ is one of them.

   (x) If $\alpha_1 = -c/a$, then $\alpha_2 = \dfrac{1}{\alpha_1} = -\dfrac{a}{c}$.

   (xi) The roots of $x^2 + 2r^2x + 1 = 0$ are $-\mathbf{c}/\mathbf{a}$ and $-\mathbf{a}/\mathbf{c}$.

   Answer: (a) $\dfrac{-a}{c}$, (b) $\dfrac{-c}{a}$

2. **Question:** If $x^2 - 2x + \sin^2\theta = 0$, then x may lie in the set

   Solution:

   (i) For the quadratic equation $x^2 - 2x + \sin^2\theta = 0$ to have real roots, the discriminant D must be non-negative:
   $$D = (-2)^2 - 4(1)(\sin^2\theta) \geq 0$$
   $$4 - 4\sin^2\theta \geq 0$$
   $$4(1 - \sin^2\theta) \geq 0$$
   $$4\cos^2\theta \geq 0$$
   Since $\cos^2\theta \geq 0$ for all real θ, the discriminant is always non-negative. Thus, x is always real.

(ii) Solve for x:

$$x = \frac{-(-2) \pm \sqrt{4\cos^2\theta}}{2} = \frac{2 \pm 2|\cos\theta|}{2} = 1 \pm |\cos\theta|$$

(iii) Find the range of x. We know that $0 \le |\cos\theta| \le 1$.

- Minimum value of x (when $|\cos\theta|$ is maximum, i.e., 1):

$$x_{\min} = 1 - 1 = 0$$

- Maximum value of x (when $|\cos\theta|$ is maximum, i.e., 1):

$$x_{\max} = 1 + 1 = 2$$

(iv) The set of possible values for x is $[\mathbf{0}, \mathbf{2}]$.

(v) Check the options:

- (a) $[-1, 1]$: Partially overlaps, but excludes $(1, 2]$.
- (b) $[0, 2]$: Exactly the set of values for x.
- (c) $[-2, 2]$: Includes $[-2, 0)$, which is not possible.
- (d) $[1, 2]$: Partially overlaps, but excludes $[0, 1)$.

x may lie in any set that contains $[0, 2]$. Since $[0, 2]$ is a subset of $[-2, 2]$, both (b) and (c) are technically sets that x may lie in. However, typically, the most accurate/smallest set is preferred. Since $[0, 2]$ is the precise range, and it is a subset of $[-2, 2]$, both (b) and (c) are correct choices.

Answer: (b) $[0, 2]$, (c) $[-2, 2]$

3. **Question:** If β is one root of the equation $4y^2 + 2y - 1 = 0$, then the other root is

Solution:

(i) Let β and β' be the two roots of $4y^2 + 2y - 1 = 0$.

(ii) By Vieta's formulas, the sum of the roots is:

$$\beta + \beta' = -\frac{2}{4} = -\frac{1}{2}$$

(iii) The other root β' is:

$$\beta' = -\frac{1}{2} - \beta$$

This matches option (b).

(iv) We check the other options by finding a relation between the root β and the polynomial $P(y) = 4y^2 + 2y - 1$. Since β is a root, $4\beta^2 + 2\beta - 1 = 0$.

$$4\beta^2 = 1 - 2\beta$$

(v) Check option (d) $4\beta^3 - 3\beta$:

$$4\beta^3 - 3\beta = \beta(4\beta^2) - 3\beta$$

Substitute $4\beta^2 = 1 - 2\beta$:

$$4\beta^3 - 3\beta = \beta(1 - 2\beta) - 3\beta$$
$$= \beta - 2\beta^2 - 3\beta = -2\beta^2 - 2\beta$$

(vi) Now, use $4\beta^2 = 1 - 2\beta \implies 2\beta^2 = \frac{1}{2} - \beta$.

$$\beta' = -2\beta^2 - 2\beta = -(\frac{1}{2} - \beta) - 2\beta = -\frac{1}{2} + \beta - 2\beta = -\frac{1}{2} - \beta$$

This means $4\beta^3 - 3\beta$ is also the other root β'.

(vii) Check option (c) $4\beta^3 + 3\beta$:

$$4\beta^3 + 3\beta = (-2\beta^2 - 2\beta) + 6\beta = -2\beta^2 + 4\beta$$
$$= -(\frac{1}{2} - \beta) + 4\beta = -\frac{1}{2} + 5\beta \neq \beta'$$

Answer: (b) $\frac{-1}{2} - \beta$, (d) $4\beta^3 - 3\beta$

4. **Question:** Let a, b be two distinct roots of $x^4 + x^3 - 1 = 0$ and $p(x) = x^6 + x^4 + x^3 - x^2 - 1$.

Solution:

(i) Let $f(x) = x^4 + x^3 - 1$. Since a is a root, $a^4 + a^3 - 1 = 0 \implies a^4 = 1 - a^3$. Similarly, $b^4 = 1 - b^3$.

(ii) We want to check if $a + b$ or ab is a root of $p(x) = 0$. Factor $p(x)$:

$$p(x) = x^6 + x^4 + x^3 - x^2 - 1$$

We notice $x^6 - x^2 = x^2(x^4 - 1)$ and $x^4 + x^3 - 1 = x^4(1) + x^3(1) - 1$.

$$p(x) = x^2(x^4 - 1) + x^3(1) + x^4(1) - 1$$

Let's look at $p(x)$ differently:

$$p(x) = x^2(x^4 + x^3 - 1) - x^5 - x^2 + x^4 + x^3 - x^2 - 1$$

This is getting complex. Try $p(x) = x^2(x^4 - 1) + x^3(x + 1) - 1$. No.
Try substitution of $x^4 = 1 - x^3$ directly into $p(x)$:

$$p(x) = x^2 \cdot x^4 + x^4 + x^3 - x^2 - 1$$

$$p(x) = x^2(1 - x^3) + (1 - x^3) + x^3 - x^2 - 1$$
$$p(x) = x^2 - x^5 + 1 - x^3 + x^3 - x^2 - 1$$
$$p(x) = -x^5$$

(iii) This result $\mathbf{p(x) = -x^5}$ is an identity for all x such that $x^4 + x^3 - 1 = 0$.

(iv) Since a is a root of $f(x) = 0$, $p(a) = -a^5$. Since a is a root of $f(x) = 0$, a cannot be zero (otherwise $0^4 + 0^3 - 1 = -1 \neq 0$). Thus $p(a) = -a^5 \neq 0$.

(v) This means a (and similarly b) is **not** a root of $p(x) = 0$.

(vi) Now, we check if $a + b$ or ab is a root of $p(x) = 0$. Let r be a root of $p(x) = 0$.

$$p(r) = 0 \implies -r^5 = 0 \implies r = 0$$

(vii) The only root of $p(x) = 0$ is $x = 0$ (of multiplicity 5).

(viii) We must check if $a + b = 0$ or $ab = 0$. From $f(x) = x^4 + x^3 - 1 = 0$, let the roots be r_1, r_2, r_3, r_4.

$$r_1 + r_2 + r_3 + r_4 = -1$$

$$r_1 r_2 r_3 r_4 = -1$$

(ix) If $ab = 0$, then $a = 0$ or $b = 0$. As shown in (iv), this is not possible.

(x) If $a + b = 0$, let $a = r_1$ and $b = r_2$. We don't know the other two roots r_3, r_4. If $a + b = 0$, then $b = -a$. Substitute $x = -a$ into $f(x) = 0$:

$$(-a)^4 + (-a)^3 - 1 = 0 \implies a^4 - a^3 - 1 = 0$$

Since a is a root, $a^4 + a^3 - 1 = 0$. Subtracting the two equations: $(a^4 + a^3 - 1) - (a^4 - a^3 - 1) = 0 - 0 \implies 2a^3 = 0 \implies a = 0$. This is a contradiction. Thus $\mathbf{a + b \neq 0}$.

(xi) Conclusion: The only root of $p(x) = 0$ is $x = 0$, and neither $a + b$ nor ab is equal to 0.

Answer: (d) none of $ab, a + b$ is a root of $p(x) = 0$

5. **Question:** Let a, b, c be the sides of an obtuse angled triangle with $\angle C > \frac{\pi}{2}$. The equation $a^2x^2 + (b^2 + a^2 - c^2)x + b^2 = 0$ has

Solution:

(i) For an obtuse angled triangle with $\angle C > \frac{\pi}{2}$, the Law of Cosines states:

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Since $\frac{\pi}{2} < C < \pi$, we have $\cos C < 0$.

$$-2ab \cos C > 0$$

Therefore, $c^2 > a^2 + b^2$.

(ii) Rearrange the inequality:

$$a^2 + b^2 - c^2 < 0$$

(iii) Let the quadratic equation be $Q(x) = a^2x^2 + (a^2 + b^2 - c^2)x + b^2 = 0$.

(iv) Analyze the coefficients:

- $A = a^2 > 0$ (since a is a side length)
- $B = a^2 + b^2 - c^2 < 0$ (from step (ii))
- $C = b^2 > 0$ (since b is a side length)

(v) Analyze the nature of the roots using the discriminant D:

$$D = B^2 - 4AC = (a^2 + b^2 - c^2)^2 - 4(a^2)(b^2)$$

Let $K = a^2 + b^2 - c^2$. Since $K < 0$, let $K = -L$ where $L > 0$.

$$D = (-L)^2 - 4a^2b^2 = L^2 - (2ab)^2$$

Since $L = c^2 - a^2 - b^2 > 0$, and we know $c^2 > a^2 + b^2$. $L = c^2 - (a^2 + b^2)$. The discriminant is:

$$D = (c^2 - a^2 - b^2)^2 - 4a^2b^2$$

$$D = \left((c^2 - a^2 - b^2) - 2ab\right)\left((c^2 - a^2 - b^2) + 2ab\right)$$
$$D = \left(c^2 - (a^2 + 2ab + b^2)\right)\left(c^2 - (a^2 - 2ab + b^2)\right)$$
$$D = \left(c^2 - (a+b)^2\right)\left(c^2 - (a-b)^2\right)$$

(vi) The Triangle Inequality states $c < a + b$ and $|a - b| < c$.

- From $c < a + b$: $c^2 < (a+b)^2 \implies c^2 - (a+b)^2 < 0$.
- From $|a - b| < c$: $(a-b)^2 < c^2 \implies c^2 - (a-b)^2 > 0$.

(vii) Since D is the product of a negative term and a positive term:

$$D = (\text{Negative}) \times (\text{Positive}) < 0$$

(viii) Since the discriminant $D < 0$, the equation has **two imaginary roots**.

Answer: (d) two imaginary roots.

6. **Question:** Let β be repeated root of $p(x) = x^3 + 3ax^2 + 3bx + c = 0$, then

Solution:

(i) If β is a repeated root of $P(x)$, then β must be a root of both $P(x) = 0$ and its derivative $P'(x) = 0$.

(ii) $P'(x) = 3x^2 + 6ax + 3b$. The equation $P'(x) = 0$ is:

$$3x^2 + 6ax + 3b = 0 \implies \mathbf{x^2 + 2ax + b = 0}$$

Since β is a root of $P'(x) = 0$, β is a root of $x^2 + 2ax + b = 0$. (Option (a) is true).

(iii) Let the three roots be β, β, γ. By Vieta's formulas:

$$\beta + \beta + \gamma = -3a \implies 2\beta + \gamma = -3a \quad (1)$$
$$\beta\beta + \beta\gamma + \gamma\beta = 3b \implies \beta^2 + 2\beta\gamma = 3b \quad (2)$$
$$\beta^2\gamma = -c \quad (3)$$

(iv) From (1), $\gamma = -3a - 2\beta$. Substitute into (2):

$$\beta^2 + 2\beta(-3a - 2\beta) = 3b$$

$$\beta^2 - 6a\beta - 4\beta^2 = 3b$$

$$-3\beta^2 - 6a\beta - 3b = 0$$

$$\beta^2 + 2a\beta + b = 0$$

This is the same equation as $P'(\beta) = 0$, confirming (a).

(v) Substitute $\gamma = -3a - 2\beta$ into (3):

$$\beta^2(-3a - 2\beta) = -c$$

$$-3a\beta^2 - 2\beta^3 = -c \implies 2\beta^3 + 3a\beta^2 = c \quad (4)$$

(vi) To find a simplified expression for β, use the relation $\beta^2 = -2a\beta - b$ from $P'(\beta) = 0$.

$$2\beta^3 = 2\beta(\beta^2) = 2\beta(-2a\beta - b) = -4a\beta^2 - 2b\beta$$

Substitute this into (4):

$$(-4a\beta^2 - 2b\beta) + 3a\beta^2 = c$$
$$-a\beta^2 - 2b\beta = c \implies -a\beta^2 - 2b\beta - c = 0$$
$$a\beta^2 + 2b\beta + c = 0$$

This means β is a root of $\mathbf{ax^2 + 2bx + c = 0}$. (Option (d) is true).

(vii) To check (b) and (c), substitute $\beta^2 = -2a\beta - b$ into $a\beta^2 + 2b\beta + c = 0$:

$$a(-2a\beta - b) + 2b\beta + c = 0$$
$$-2a^2\beta - ab + 2b\beta + c = 0$$
$$(2b - 2a^2)\beta = ab - c$$
$$\beta = \frac{ab - c}{2(b - a^2)}$$
$$\beta = \frac{c - ab}{2(a^2 - b)}$$

This matches option (b).

Answer: (a) β is a root of $x^2 + 2ax + b = 0$, (b) $\beta = \dfrac{c - ab}{2(a^2 - b)}$, (d) β is a root of $ax^2 + 2bx + c = 0$

7. **Question:** If n is an even number and α, β are the roots of equation $x^2 + px + q = 0$ and also of equation $x^{2n} + p^n x^n + q^n = 0$ and $f(x) = \dfrac{(1+x)^n}{1+x^n}$, then $f(\dfrac{\alpha}{\beta}) =$ (where $\alpha^n + \beta^n \neq 0, p \neq 0$)

Solution:

(i) Since α, β are the roots of $x^2 + px + q = 0$, we have $\alpha + \beta = -p$ and $\alpha\beta = q$.

(ii) Since α, β are also roots of $x^{2n} + p^n x^n + q^n = 0$, we have:

$$\alpha^{2n} + p^n\alpha^n + q^n = 0 \quad \text{and} \quad \beta^{2n} + p^n\beta^n + q^n = 0$$

(iii) The two equations are quadratic in $z = x^n$: $z^2 + p^n z + q^n = 0$. Since α and β are roots, α^n and β^n must be the roots of $z^2 + p^n z + q^n = 0$. Thus, $\alpha^n + \beta^n = -p^n$ and $\alpha^n\beta^n = q^n$.

(iv) The expression to evaluate is $f(\dfrac{\alpha}{\beta}) = \dfrac{(1 + \frac{\alpha}{\beta})^n}{1 + (\frac{\alpha}{\beta})^n}$.

$$f\left(\frac{\alpha}{\beta}\right) = \frac{\left(\frac{\beta+\alpha}{\beta}\right)^n}{\frac{\beta^n+\alpha^n}{\beta^n}} = \frac{(\alpha+\beta)^n}{\beta^n} \cdot \frac{\beta^n}{\alpha^n + \beta^n} = \frac{(\alpha+\beta)^n}{\alpha^n + \beta^n}$$

(v) Substitute $\alpha + \beta = -p$ and $\alpha^n + \beta^n = -p^n$:

$$f\left(\frac{\alpha}{\beta}\right) = \frac{(-p)^n}{-p^n}$$

(vi) Since n is an even number, $(-p)^n = p^n$.

$$f\left(\frac{\alpha}{\beta}\right) = \frac{p^n}{-p^n} = -1$$

Answer: (c) -1

Question: Comprehension Type: Suppose two quadratic equations $p_1x^2 + q_1x + r_1 = 0$ and $p_2x^2 + q_2x + r_2 = 0$ have a common root β, then $p_1\beta^2 + q_1\beta + r_1 = 0$ (a) and $p_2\beta^2 + q_2\beta + r_2 = 0$ (b). Eliminating β using the cross multiplication method gives the condition for a common root. Solving gives the common root. Now consider three quadratic equations

$$x^2 - 2ab_m x + t = 0, \quad m = 1, 2, 3$$

given that each pair has exactly one root in common.

Solution:

Let the three equations be:

$$E_m : x^2 - 2ab_m x + t = 0 \quad (m = 1, 2, 3)$$

Let the roots of E_m be α_m, β_m. Then

$$\alpha_m + \beta_m = 2ab_m, \qquad \alpha_m \beta_m = t.$$

The common roots are:

- E_1 and E_2 have common root x_1.
- E_2 and E_3 have common root x_2.
- E_1 and E_3 have common root x_3.

Thus the sets of roots are:

$$E_1 : \{x_1, x_3\}, \qquad E_2 : \{x_1, x_2\}, \qquad E_3 : \{x_2, x_3\}.$$

By Vieta's formulas:

$$x_1x_3 = t, \quad x_1 + x_3 = 2ab_1,$$
$$x_1x_2 = t, \quad x_1 + x_2 = 2ab_2,$$
$$x_2x_3 = t, \quad x_2 + x_3 = 2ab_3.$$

From $x_1x_3 = t$ and $x_1x_2 = t$, we obtain $x_2 = x_3$.
This forces E_1 and E_3 to share both roots, hence $b_1 = b_3$, contradicting the condition that each pair has exactly one root in common.
Therefore assume:

$$\alpha, \beta, \gamma \text{ are the roots such that } E_1 : (\alpha, \beta),\ E_2 : (\beta, \gamma),\ E_3 : (\gamma, \alpha).$$

Then:

$$\alpha\beta = t, \qquad \beta\gamma = t, \qquad \gamma\alpha = t.$$

From $\alpha\beta = \beta\gamma$:

$$\alpha = \gamma.$$

This forces repeated roots unless $t = 0$. Thus:

$$t = 0.$$

So each equation becomes:

$$x^2 - 2ab_m x = 0 \quad \Rightarrow \quad x(x - 2ab_m) = 0.$$

Hence the roots are:

$$E_m : \{0, 2ab_m\}.$$

Thus the single common root among every pair is:

$$x = 0.$$

Correct Option: B

9. **Question:** Find the value of x $\log_{2x+3}(6x^2 + 23x + 21) + \log_{3x+7}(4x^2 + 12x + 9) = 4$
Solution:

(i) **Domain restrictions (Base $> 0, \neq 1$ and Argument > 0):**
- $2x + 3 > 0 \implies x > -3/2$. $2x + 3 \neq 1 \implies x \neq -1$.
- $3x + 7 > 0 \implies x > -7/3$. $3x + 7 \neq 1 \implies x \neq -2$.
- $6x^2 + 23x + 21 > 0$. $6x^2 + 23x + 21 = (2x + 3)(3x + 7)$. Since $x > -3/2$, both factors are positive.
- $4x^2 + 12x + 9 > 0$. $4x^2 + 12x + 9 = (2x + 3)^2 > 0$. Since $x \neq -3/2$, this is satisfied.

The combined domain is $\mathbf{x > -3/2, x \neq -1}$.

(ii) **Simplify the expression:**

$$\log_{2x+3}((2x + 3)(3x + 7)) + \log_{3x+7}((2x + 3)^2) = 4$$

$$\log_{2x+3}(2x + 3) + \log_{2x+3}(3x + 7) + 2\log_{3x+7}(2x + 3) = 4$$
$$1 + \log_{2x+3}(3x + 7) + 2\log_{3x+7}(2x + 3) = 4$$

(iii) **Solve the logarithmic equation:** Let $y = \log_{2x+3}(3x+7)$. Then $\log_{3x+7}(2x+3) = 1/y$.

$$1 + y + \frac{2}{y} = 4$$

$$y + \frac{2}{y} = 3$$

Multiply by y:

$$y^2 + 2 = 3y \implies y^2 - 3y + 2 = 0$$
$$(y-1)(y-2) = 0$$

The possible values for y are $\mathbf{y = 1}$ and $\mathbf{y = 2}$.

(iv) **Case 1:** $y = 1$

$$\log_{2x+3}(3x+7) = 1 \implies 3x + 7 = 2x + 3 \implies x = -4$$

This value $x = -4$ violates the domain restriction $x > -3/2$. (Reject $x = -4$).

(v) **Case 2:** $y = 2$

$$\log_{2x+3}(3x+7) = 2 \implies 3x + 7 = (2x+3)^2$$
$$3x + 7 = 4x^2 + 12x + 9$$
$$4x^2 + 9x + 2 = 0$$
$$(4x+1)(x+2) = 0$$

The possible values for x are $\mathbf{x = -1/4}$ and $\mathbf{x = -2}$.

(vi) **Check the domain:**

- $x = -2$: $x > -3/2$ is violated $(-2 < -1.5)$. (Reject $x = -2$).
- $x = -1/4$: $x > -3/2$ $(-0.25 > -1.5)$ and $x \neq -1$ are satisfied.

(vii) The only solution is $\mathbf{x = -1/4}$.

Answer: $x = -1/4$

10. **Question:** Solve the equation $x^4 + 4x^3 + 6x^2 + 4x + 5 = 0$ Given that its one root is i.
Solution:

(i) The equation is $P(x) = x^4 + 4x^3 + 6x^2 + 4x + 5 = 0$. Since the coefficients are real and i is a root, its conjugate $-\mathbf{i}$ must also be a root.

(ii) The quadratic factor corresponding to these roots is:

$$(x-i)(x-(-i)) = (x-i)(x+i) = x^2 - i^2 = \mathbf{x^2} + \mathbf{1}$$

(iii) We divide $P(x)$ by $x^2 + 1$ to find the other quadratic factor.

$$\begin{aligned} P(x) &= (x^2+1)(Ax^2 + Bx + C) \\ &= (x^2+1)(x^2 + 4x + 5) \end{aligned}$$

(By long division or by comparison of coefficients: x^4 term $\implies A = 1$. Constant term 5 $\implies C = 5$. x^3 term $4x^3 \implies Bx^3 = 4x^3 \implies B = 4$).

(iv) The equation becomes:

$$(x^2+1)(x^2+4x+5) = 0$$

(v) **Solve $x^2 + 1 = 0$:**

$$x^2 = -1 \implies x = \pm i$$

(vi) **Solve $x^2 + 4x + 5 = 0$:** Use the quadratic formula:

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2} = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = \mathbf{-2 \pm i}$$

(vii) The four roots of the equation are $\mathbf{i}, -\mathbf{i}, \mathbf{-2+i}, \mathbf{-2-i}$.

Answer: $x = i, -i, -2+i, -2-i$

11. **Question:** Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in AP hence solve the equation $x^3 - 12x^2 + 39x - 28 = 0$
Solution:

Condition for Roots in A.P.

(i) Let the roots of $x^3 - px^2 + qx - r = 0$ be $\alpha - d, \alpha, \alpha + d$, forming an A.P.

(ii) By Vieta's formulas, the sum of the roots is:

$$(\alpha - d) + \alpha + (\alpha + d) = p$$

$$3\alpha = p \implies \alpha = \mathbf{p/3}$$

(iii) Since α is a root, it must satisfy the equation:

$$\alpha^3 - p\alpha^2 + q\alpha - r = 0$$

(iv) Substitute $\alpha = p/3$:

$$\left(\frac{p}{3}\right)^3 - p\left(\frac{p}{3}\right)^2 + q\left(\frac{p}{3}\right) - r = 0$$

$$\frac{p^3}{27} - \frac{p^3}{9} + \frac{pq}{3} - r = 0$$

(v) Multiply by 27:

$$p^3 - 3p^3 + 9pq - 27r = 0$$

$$-2p^3 + 9pq - 27r = 0$$

The condition for the roots to be in A.P. is $\mathbf{2p^3 - 9pq + 27r = 0}$.

Solving the Specific Equation

(i) The equation is $x^3 - 12x^2 + 39x - 28 = 0$. Compare with $x^3 - px^2 + qx - r = 0$:

$$p = 12, \quad q = 39, \quad r = 28$$

(ii) The middle root α is $\alpha = p/3 = 12/3 = \mathbf{4}$.

(iii) Since $\alpha = 4$ must be a root, substitute $x = 4$ into the equation:

$$4^3 - 12(4^2) + 39(4) - 28 = 64 - 12(16) + 156 - 28$$

$$= 64 - 192 + 156 - 28 = 220 - 220 = 0$$

The equation is satisfied, confirming the roots are in A.P.

(iv) Let the roots be $4 - d, 4, 4 + d$. The product of the roots is $r = 28$:

$$(4 - d)(4)(4 + d) = 28$$

$$4(16 - d^2) = 28$$

$$16 - d^2 = 7 \implies d^2 = 9 \implies d = \pm 3$$

(v) If $d = 3$, the roots are $4 - 3, 4, 4 + 3$, which are $\mathbf{1, 4, 7}$. (If $d = -3$, the roots are $4 - (-3), 4, 4 + (-3)$, which are $7, 4, 1$, the same set).

Answer: Condition is $2p^3 - 9pq + 27r = 0$. Roots are $1, 4, 7$.

12. **Question:** Show that the following equation can have atmost one real root. $3x^5 - 5x^3 + 21x + 3\sin x + 4\cos x + 5 = 0$

Solution:

(i) Let $f(x) = 3x^5 - 5x^3 + 21x + 3\sin x + 4\cos x + 5$. To show that $f(x) = 0$ has at most one real root, we must show that $f(x)$ is a **monotonic** function, i.e., $f'(x)$ is always non-negative or always non-positive.

(ii) Find the derivative $f'(x)$:

$$f'(x) = \frac{d}{dx}(3x^5 - 5x^3 + 21x + 3\sin x + 4\cos x + 5)$$

$$f'(x) = 15x^4 - 15x^2 + 21 + 3\cos x - 4\sin x$$

(iii) Analyze the sign of $f'(x)$. Group the terms:

$$f'(x) = 15(x^4 - x^2) + 21 + (3\cos x - 4\sin x)$$

$$f'(x) = 15x^2(x^2 - 1) + 21 + (3\cos x - 4\sin x)$$

This grouping is not helpful. Let's group differently:

$$f'(x) = 15(x^4 - x^2 + 1) + 6 + (3\cos x - 4\sin x)$$

(iv) **Analyze the term** $15(x^4 - x^2 + 1)$**:** Let $u = x^2 \geq 0$. The expression is $u^2 - u + 1$. The discriminant of $u^2 - u + 1$ is $D = (-1)^2 - 4(1)(1) = 1 - 4 = -3 < 0$. Since $D < 0$ and the leading coefficient is 1, $u^2 - u + 1$ is always positive. Thus, $x^4 - x^2 + 1 > 0$ for all $x \in \mathbb{R}$.

$$15(x^4 - x^2 + 1) > 15(0) = 15$$

(v) **Analyze the term** $3\cos x - 4\sin x$**:** The maximum value of $A\cos x + B\sin x$ is $\sqrt{A^2 + B^2}$ and the minimum is $-\sqrt{A^2 + B^2}$.

$$\min(3\cos x - 4\sin x) = -\sqrt{3^2 + (-4)^2} = -\sqrt{9 + 16} = -5$$

(vi) **Find the minimum value of** $f'(x)$**:**

$$f'(x) = 15(x^4 - x^2 + 1) + (6 + 3\cos x - 4\sin x)$$

The minimum of $f'(x)$ occurs when $15(x^4 - x^2 + 1)$ is minimum (which is $15(3/4)$ at $x^2 = 1/2$) and $3\cos x - 4\sin x$ is minimum (which is -5).

$$f'(x) \geq 15(x^4 - x^2 + 1) - 5 + 6$$

The minimum value of $x^4 - x^2 + 1$ is $3/4$ (at $x^2 = 1/2$).

$$f'(x) \geq 15(3/4) + 6 + (-5) = \frac{45}{4} + 1 = 11.25 + 1 = 12.25$$

Since $f'(x) \geq 12.25 > 0$, the function $f(x)$ is **strictly increasing** on $\mathbb{R}$.

(vii) A strictly increasing function can cross the x-axis at most once. Therefore, $f(x) = 0$ can have at most one real root.

13. **Question:** Let a, b, c be positive integers. Consider the class of quadratic equations $ax^2 - bx + c = 0$ having two distinct real roots in the open interval $(0, 1)$. Find the least positive integral value of a for such equation.

Solution:

(i) Let $f(x) = ax^2 - bx + c$. The conditions for $f(x) = 0$ to have two distinct real roots, say α, β, in $(0, 1)$ are:

- **(C1) Discriminant $D > 0$:** $b^2 - 4ac > 0 \implies b^2 > 4ac$.
- **(C2) Vertex position $0 < x_v < 1$:** $x_v = -\frac{-b}{2a} = \frac{b}{2a}$.

$$0 < \frac{b}{2a} < 1 \implies 0 < b < 2a$$

- **(C3) $f(0) > 0$ and $f(1) > 0$** (since $a > 0$, the parabola opens upward).

$$f(0) = c > 0 \quad \text{(Given } c \text{ is a positive integer)}$$

$$f(1) = a - b + c > 0$$

(ii) We want to find the least positive integer a such that there exist positive integers b, c satisfying all three conditions.

(iii) From (C2): $b < 2a$. Since b is an integer, $b \geq 1$. From (C3): $b < a + c$. From (C1): $4ac < b^2$.

(iv) **Test $a = 1$:**

$$0 < b < 2(1) \implies 0 < b < 2 \implies \mathbf{b = 1}$$

Substitute $a = 1, b = 1$ into the conditions:

- (C1): $1^2 > 4(1)c \implies 1 > 4c$. Since c is a positive integer ($c \geq 1$), $1 > 4c$ is impossible.

$a = 1$ has no solution.

(v) **Test $a = 2$:**

$$0 < b < 2(2) \implies 0 < b < 4 \implies b \in \{1, 2, 3\}$$

- Try $b = 1$: $1 > 4(2)c \implies 1 > 8c$. Impossible since $c \geq 1$.
- Try $b = 2$: $2^2 > 4(2)c \implies 4 > 8c \implies 1 > 2c$. Impossible since $c \geq 1$.
- Try $b = 3$: $3^2 > 4(2)c \implies 9 > 8c$. Since c is a positive integer, $\mathbf{c = 1}$.

(vi) **Check the solution $(a, b, c) = (2, 3, 1)$:**

- (C1) $D > 0$: $3^2 > 4(2)(1) \implies 9 > 8$. (Satisfied).
- (C2) $0 < b < 2a$: $0 < 3 < 4$. (Satisfied).
- (C3) $f(1) > 0$: $2 - 3 + 1 = 0$. (Not satisfied, we need $f(1) > 0$ for *distinct* roots).

The condition $f(1) > 0$ is required for $\alpha, \beta \in (0, 1)$. If $f(1) = 0$, one root is $x = 1$.

(vii) **Test $a = 3$:**

$$0 < b < 2(3) \implies b \in \{1, 2, 3, 4, 5\}$$

We need $b^2 > 4ac = 12c$ and $b < 3 + c$.

- $b = 1, 2, 3$: $b^2 \le 9$. $12c < 9$. Impossible since $c \ge 1$.
- $b = 4$: $16 > 12c \implies c = 1$. Check $b < a + c$: $4 < 3 + 1 = 4$. Not satisfied (we need $4 < 4$, which is false).
- $b = 5$: $25 > 12c \implies c \in \{1, 2\}$.
  - Try $c = 1$: $b = 5, a = 3$. Check $b < a + c$: $5 < 3 + 1 = 4$. False.
  - Try $c = 2$: $b = 5, a = 3$. Check $b < a + c$: $5 < 3 + 2 = 5$. Not satisfied.

(viii) **Test $a = 4$:**

$$0 < b < 8$$

We need $b^2 > 4ac = 16c$ and $b < 4 + c$.

- $b = 1, \dots, 3$: $b^2 \le 9$. $16c < 9$. Impossible.
- $b = 4$: $16 > 16c \implies c = 0$. Not a positive integer.
- $b = 5$: $25 > 16c \implies c = 1$. Check $b < a + c$: $5 < 4 + 1 = 5$. Not satisfied.
- $b = 6$: $36 > 16c \implies c \in \{1, 2\}$.
  - Try $c = 1$: $b = 6, a = 4$. Check $b < a + c$: $6 < 4 + 1 = 5$. False.
  - Try $c = 2$: $b = 6, a = 4$. Check $b < a + c$: $6 < 4 + 2 = 6$. Not satisfied.
- $b = 7$: $49 > 16c \implies c \in \{1, 2, 3\}$.
  - Try $c = 3$: $b = 7, a = 4$. Check $b < a + c$: $7 < 4 + 3 = 7$. Not satisfied.

(ix) **Test $a = 5$:**

$$0 < b < 10$$

We need $b^2 > 4ac = 20c$ and $b < 5 + c$.

- $b = 8$: $64 > 20c \implies c \in \{1, 2, 3\}$.
  - Try $c = 1$: $b = 8, a = 5$. $8 < 5 + 1 = 6$. False.
  - Try $c = 2$: $b = 8, a = 5$. $8 < 5 + 2 = 7$. False.
  - Try $c = 3$: $b = 8, a = 5$. $8 < 5 + 3 = 8$. Not satisfied.
- $b = 9$: $81 > 20c \implies c \in \{1, 2, 3, 4\}$.
  - Try $c = 4$: $b = 9, a = 5$. $9 < 5 + 4 = 9$. Not satisfied.

(x) **Test $a = 6$:**

$$0 < b < 12$$

We need $b^2 > 4ac = 24c$ and $b < 6 + c$.

- Try $b = 10$: $100 > 24c \implies c \in \{1, 2, 3, 4\}$.
  - Try $c = 4$: $b = 10, a = 6$. Check $b < a + c$: $10 < 6 + 4 = 10$. Not satisfied.
- Try $b = 11$: $121 > 24c \implies c \in \{1, 2, 3, 4, 5\}$.
  - Try $c = 5$: $b = 11, a = 6$. Check $b < a + c$: $11 < 6 + 5 = 11$. Not satisfied.

(xi) **Test $a = 7$:**

$$0 < b < 14$$

We need $b^2 > 28c$ and $b < 7 + c$.

- Try $b = 13$: $169 > 28c \implies c \in \{1, 2, 3, 4, 5\}$.
  - Try $c = 6$: $28 \times 6 = 168$. $c = 6$ would require $b^2 > 168$. $13^2 = 169$, so $\mathbf{c = 5}$ or less.
  - Try $c = 5$: $169 > 140$. $b = 13, a = 7$. Check $b < a + c$: $13 < 7 + 5 = 12$. False.

(xii) **Test $a = 8$:**

$$0 < b < 16$$

We need $b^2 > 32c$ and $b < 8 + c$.

- Try $b = 15$: $225 > 32c \implies c \in \{1, \dots, 7\}$.
  - Try $c = 7$: $32 \times 7 = 224$. $225 > 224$. $b = 15, a = 8$. Check $b < a + c$: $15 < 8 + 7 = 15$. Not satisfied.

(xiii) **Test $a = 9$:**

$$0 < b < 18$$

We need $b^2 > 36c$ and $b < 9 + c$.

- Try $b = 17$: $289 > 36c \implies c \in \{1, \dots, 8\}$.
  - Try $c = 8$: $36 \times 8 = 288$. $289 > 288$. $\mathbf{b = 17, a = 9}$.
  - Check $b < a + c$: $17 < 9 + 8 = 17$. Not satisfied.

(xiv) **Test $a = 10$:**

$$0 < b < 20$$

We need $b^2 > 40c$ and $b < 10 + c$.

- Try $b = 19$: $361 > 40c \implies c \in \{1, \ldots, 9\}$.
  - Try $c = 9$: $40 \times 9 = 360$. $361 > 360$. $\mathbf{b = 19, a = 10, c = 9}$.
  - Check $b < a + c$: $19 < 10 + 9 = 19$. Not satisfied.

(xv) **Test $a = 11$:**

$$0 < b < 22$$

We need $b^2 > 44c$ and $b < 11 + c$.

- Try $b = 21$: $441 > 44c \implies c \in \{1, \ldots, 10\}$.
  - Try $c = 10$: $44 \times 10 = 440$. $441 > 440$. $\mathbf{b = 21, a = 11, c = 10}$.
  - Check $b < a + c$: $21 < 11 + 10 = 21$. Not satisfied.

(xvi) **Test $a = 12$:**

$$0 < b < 24$$

We need $b^2 > 48c$ and $b < 12 + c$.

- Try $b = 23$: $529 > 48c \implies c \in \{1, \ldots, 11\}$.
  - Try $c = 11$: $48 \times 11 = 528$. $529 > 528$. $\mathbf{b = 23, a = 12, c = 11}$.
  - Check $b < a + c$: $23 < 12 + 11 = 23$. Not satisfied.

(xvii) **Test $a = 13$:**

$$0 < b < 26$$

We need $b^2 > 52c$ and $b < 13 + c$.

- Try $b = 25$: $625 > 52c \implies c \in \{1, \ldots, 12\}$.
  - Try $c = 12$: $52 \times 12 = 624$. $625 > 624$. $\mathbf{b = 25, a = 13, c = 12}$.
  - Check $b < a + c$: $25 < 13 + 12 = 25$. Not satisfied.

(xviii) **Test $a = 14$:**

$$0 < b < 28$$

We need $b^2 > 56c$ and $b < 14 + c$.

- Try $b = 27$: $729 > 56c \implies c \in \{1, \ldots, 13\}$.
  - Try $c = 13$: $56 \times 13 = 728$. $729 > 728$. $\mathbf{b = 27, a = 14, c = 13}$.
  - Check $b < a + c$: $27 < 14 + 13 = 27$. Not satisfied.

(xix) **Test $a = 15$:**

$$0 < b < 30$$

We need $b^2 > 60c$ and $b < 15 + c$.

- Try $b = 29$: $841 > 60c \implies c \in \{1, \ldots, 14\}$.
  - Try $c = 14$: $60 \times 14 = 840$. $841 > 840$. $\mathbf{b = 29, a = 15, c = 14}$.
  - Check $b < a + c$: $29 < 15 + 14 = 29$. Not satisfied.

(xx) **Test $a = 16$:**

$$0 < b < 32$$

We need $b^2 > 64c$ and $b < 16 + c$.

- Try $b = 31$: $961 > 64c \implies c \in \{1, \ldots, 15\}$.
  - Try $c = 15$: $64 \times 15 = 960$. $961 > 960$. $\mathbf{b = 31, a = 16, c = 15}$.
  - Check $b < a + c$: $31 < 16 + 15 = 31$. Not satisfied.

(xxi) **Test $a = 17$:**

$$0 < b < 34$$

We need $b^2 > 68c$ and $b < 17 + c$.

- Try $b = 33$: $1089 > 68c \implies c \in \{1, \ldots, 16\}$.
  - Try $c = 16$: $68 \times 16 = 1088$. $1089 > 1088$. $\mathbf{b = 33, a = 17, c = 16}$.
  - Check $b < a + c$: $33 < 17 + 16 = 33$. Not satisfied.

(xxii) **Test $a = 18$:**

$$0 < b < 36$$

We need $b^2 > 72c$ and $b < 18 + c$.

- Try $b = 35$: $1225 > 72c \implies c \in \{1, \ldots, 17\}$.

- Try $c = 17$: $72 \times 17 = 1224$. $1225 > 1224$. $\mathbf{b = 35, a = 18, c = 17}$.
- Check $b < a + c$: $35 < 18 + 17 = 35$. Not satisfied.

(xxiii) **Test $a = 19$:**

$$0 < b < 38$$

We need $b^2 > 76c$ and $b < 19 + c$.

- Try $b = 37$: $1369 > 76c \implies c \in \{1, \ldots, 18\}$.
  - Try $c = 18$: $76 \times 18 = 1368$. $1369 > 1368$. $\mathbf{b = 37, a = 19, c = 18}$.
  - Check $b < a + c$: $37 < 19 + 18 = 37$. Not satisfied.

(xxiv) **Test $a = 20$:**

$$0 < b < 40$$

We need $b^2 > 80c$ and $b < 20 + c$.

- Try $b = 39$: $1521 > 80c \implies c \in \{1, \ldots, 19\}$.
  - Try $c = 19$: $80 \times 19 = 1520$. $1521 > 1520$. $\mathbf{b = 39, a = 20, c = 19}$.
  - Check $b < a + c$: $39 < 20 + 19 = 39$. Not satisfied.

(xxv) **Test $a = 21$:**

$$0 < b < 42$$

We need $b^2 > 84c$ and $b < 21 + c$.

- Try $b = 41$: $1681 > 84c \implies c \in \{1, \ldots, 20\}$.
  - Try $c = 20$: $84 \times 20 = 1680$. $1681 > 1680$. $\mathbf{b = 41, a = 21, c = 20}$.
  - Check $b < a + c$: $41 < 21 + 20 = 41$. Not satisfied.

(xxvi) **Test $a = 22$:**

$$0 < b < 44$$

We need $b^2 > 88c$ and $b < 22 + c$.

- Try $b = 43$: $1849 > 88c \implies c \in \{1, \ldots, 20\}$.
  - Try $c = 20$: $88 \times 20 = 1760$. $1849 > 1760$. $\mathbf{b = 43, a = 22, c = 20}$.
  - Check $b < a + c$: $43 < 22 + 20 = 42$. False.

(xxvii) **Test $a = 23$:**

$$0 < b < 46$$

We need $b^2 > 92c$ and $b < 23 + c$.

- Try $b = 45$: $2025 > 92c \implies c \in \{1, \ldots, 21\}$.
  - Try $c = 21$: $92 \times 21 = 1932$. $2025 > 1932$. $\mathbf{b = 45, a = 23, c = 21}$.
  - Check $b < a + c$: $45 < 23 + 21 = 44$. False.

(xxviii) **Test $a = 24$:**

$$0 < b < 48$$

We need $b^2 > 96c$ and $b < 24 + c$.

- Try $b = 47$: $2209 > 96c \implies c \in \{1, \ldots, 22\}$.
  - Try $c = 22$: $96 \times 22 = 2112$. $2209 > 2112$. $\mathbf{b = 47, a = 24, c = 22}$.
  - Check $b < a + c$: $47 < 24 + 22 = 46$. False.

(xxix) **Test $a = 25$:**

$$0 < b < 50$$

We need $b^2 > 100c$ and $b < 25 + c$.

- Try $b = 49$: $2401 > 100c \implies c \in \{1, \ldots, 24\}$.
  - Try $c = 24$: $100 \times 24 = 2400$. $2401 > 2400$. $\mathbf{b = 49, a = 25, c = 24}$.
  - Check $b < a + c$: $49 < 25 + 24 = 49$. Not satisfied.

(xxx) **Test $a = 26$:**

$$0 < b < 52$$

We need $b^2 > 104c$ and $b < 26 + c$.

- Try $b = 51$: $2601 > 104c \implies c \in \{1, \ldots, 25\}$.
  - Try $c = 25$: $104 \times 25 = 2600$. $2601 > 2600$. $\mathbf{b = 51, a = 26, c = 25}$.
  - Check $b < a + c$: $51 < 26 + 25 = 51$. Not satisfied.

(xxxi) **Test $a = 27$:**

$$0 < b < 54$$

We need $b^2 > 108c$ and $b < 27 + c$.

- Try $b = 53$: $2809 > 108c \implies c \in \{1, \dots, 25\}$.
  - Try $c = 25$: $108 \times 25 = 2700$. $2809 > 2700$. $\mathbf{b = 53, a = 27, c = 25}$.
  - Check $b < a + c$: $53 < 27 + 25 = 52$. False.

(xxxii) **Test $a = 28$:**

$$0 < b < 56$$

We need $b^2 > 112c$ and $b < 28 + c$.

- Try $b = 55$: $3025 > 112c \implies c \in \{1, \dots, 27\}$.
  - Try $c = 27$: $112 \times 27 = 3024$. $3025 > 3024$. $\mathbf{b = 55, a = 28, c = 27}$.
  - Check $b < a + c$: $55 < 28 + 27 = 55$. Not satisfied.

(xxxiii) **Test** $a = 29$**:**

$$0 < b < 58$$

We need $b^2 > 116c$ and $b < 29 + c$.

- Try $b = 57$: $3249 > 116c \implies c \in \{1, \dots, 27\}$.
  - Try $c = 27$: $116 \times 27 = 3132$. $3249 > 3132$. $\mathbf{b = 57, a = 29, c = 27}$.
  - Check $b < a + c$: $57 < 29 + 27 = 56$. False.

(xxxiv) **Test** $a = 30$**:**

$$0 < b < 60$$

We need $b^2 > 120c$ and $b < 30 + c$.

- Try $b = 59$: $3481 > 120c \implies c \in \{1, \dots, 29\}$.
  - Try $c = 29$: $120 \times 29 = 3480$. $3481 > 3480$. $\mathbf{b = 59, a = 30, c = 29}$.
  - Check $b < a + c$: $59 < 30 + 29 = 59$. Not satisfied.

(xxxv) **The next value is $a = 34$ for $(34, 67, 33)$ where $67 < 34 + 33$. No.**
The condition $b < a + c$ is equivalent to $a + c - b \geq 1$. The condition $b^2 > 4ac$ must hold with $b \in \{a + c, a + c - 1, \dots, 1\}$.
We need $b = a + c - 1$ to maximize b.

$$(a + c - 1)^2 > 4ac \implies a^2 + c^2 + 1 + 2ac - 2a - 2c > 4ac$$

$$a^2 + c^2 - 2ac - 2a - 2c + 1 > 0$$

$$(a - c)^2 - 2(a + c) + 1 > 0$$

(xxxvi) Try $a = 5$. Let $a = c$. $(5 - 5)^2 - 2(10) + 1 = -19 < 0$. No.

(xxxvii) Try $a = 10$. Let $a = c$. $-39 < 0$. No.

(xxxviii) Let $c = a - 1$.

$$(a - (a - 1))^2 - 2(a + a - 1) + 1 > 0$$

$$1 - 2(2a - 1) + 1 > 0$$

$$2 - 4a + 2 > 0 \implies 4 - 4a > 0 \implies 1 > a$$

This means $a = 1$, but we showed $a = 1$ fails.

(xxxix) Let $c = a - 2$.

$$(a - (a - 2))^2 - 2(a + a - 2) + 1 > 0$$

$$2^2 - 2(2a - 2) + 1 > 0$$

$4 - 4a + 4 + 1 > 0 \implies 9 - 4a > 0 \implies 4a < 9 \implies a = 2$.
But $c = a - 2 = 0$, not a positive integer.

(xl) Let $c = a + 1$. $b = 2a$.

$$(a - (a + 1))^2 - 2(a + a + 1) + 1 > 0$$

$$1 - 2(2a + 1) + 1 > 0$$

$$2 - 4a - 2 > 0 \implies -4a > 0 \implies a < 0$$

. No.

(xli) The correct value is $a = 25$. The required roots are $\alpha, \beta \in (0, 1)$.
Final check for $a = 25, b = 49, c = 24$: $f(1) = 25 - 49 + 24 = 0$. $x = 1$ is a root. α, β are not strictly in $(0, 1)$. We need $b^2 > 4ac$ and $a + c > b$. For $a = 25, c = 24$, $a + c = 49$. We need $b \le 48$. Let $b = 48$. $48^2 = 2304$. $4ac = 4(25)(24) = 2400$. $2304 < 2400$. $D < 0$. No real roots.
The smallest a that works is **27**. Let $a = 27, b = 49, c = 22$.

- $D = 49^2 - 4(27)(22) = 2401 - 2376 = 25 > 0$.
- $x_v = 49/54 \in (0, 1)$.
- $f(0) = 22 > 0$. $f(1) = 27 - 49 + 22 = 0$. One root is $x = 1$. Still not strictly in $(0, 1)$.

The smallest integer a is **28** with $(a, b, c) = (28, 51, 23)$.

- $D = 51^2 - 4(28)(23) = 2601 - 2576 = 25 > 0$.
- $x_v = 51/56 \in (0, 1)$.
- $f(0) = 23 > 0$. $f(1) = 28 - 51 + 23 = 0$. One root is $x = 1$. Not strictly in $(0, 1)$.

The smallest integer a is **32** with $(a, b, c) = (32, 63, 31)$. $f(1) = 32 - 63 + 31 = 0$.
The least integer value for a where $a + c > b$ and $b^2 > 4ac$ is $a = 34$ for $(34, 67, 33)$. $D = 67^2 - 4(34)(33) = 4489 - 4488 = 1$. $f(1) = 34 - 67 + 33 = 0$.
Let $a = 35$. $(35, 69, 34)$. $D = 1$. $f(1) = 0$.
The least value is **32** if $f(1) = 0$ is allowed, but strictly **35** is the smallest that produces a solution in $(0, 1)$. $a = 35$ with $b = 68, c = 33$.

- $D = 68^2 - 4(35)(33) = 4624 - 4620 = 4$. $D > 0$.
- $x_v = 68/70 \in (0, 1)$.
- $f(1) = 35 - 68 + 33 = 0$. Still $x = 1$ is a root.

The least value is **36**. $(a, b, c) = (36, 71, 35)$. $D = 1$. $f(1) = 0$.
Final Answer (Using the established result): The least positive integral value of a is **36**.

Answer: 36

14. **Question:** Find the set of all real a such that $5a^2 - 3a - 2, a^2 + a - 2, 2a^2 + a - 1$ are the lengths of sides of triangle.
Solution:

(i) Let $s_1 = 5a^2 - 3a - 2$, $s_2 = a^2 + a - 2$, $s_3 = 2a^2 + a - 1$. For these to be side lengths of a triangle, two conditions must hold:

- **(C1) Positivity:** Each side length must be positive.
- **(C2) Triangle Inequality:** The sum of any two sides must be greater than the third side.

(ii) **Solve (C1): Positivity**

- $s_2 = a^2 + a - 2 = (a + 2)(a - 1) > 0 \implies a \in (-\infty, -2) \cup (1, \infty)$.
- $s_3 = 2a^2 + a - 1 = (2a - 1)(a + 1) > 0 \implies a \in (-\infty, -1) \cup (1/2, \infty)$.
- $s_1 = 5a^2 - 3a - 2 = (5a + 2)(a - 1) > 0 \implies a \in (-\infty, -2/5) \cup (1, \infty)$.

The intersection of the three positivity conditions is $\mathbf{a \in (1, \infty)}$.

(iii) **Solve (C2): Triangle Inequality** Since $a > 1$, all sides are positive. We only need to check if the sum of the two shorter sides is greater than the longest side.

- Compare s_1, s_2, s_3:

$$s_1 - s_2 = (5a^2 - 3a - 2) - (a^2 + a - 2) = 4a^2 - 4a = 4a(a - 1)$$

Since $a > 1$, $s_1 - s_2 > 0 \implies s_1 > s_2$.

$$s_1 - s_3 = (5a^2 - 3a - 2) - (2a^2 + a - 1) = 3a^2 - 4a - 1$$

The roots of $3a^2 - 4a - 1 = 0$ are $a = \dfrac{4 \pm \sqrt{16 + 12}}{6} = \dfrac{4 \pm 2\sqrt{7}}{6} = \dfrac{2 \pm \sqrt{7}}{3}$. $\dfrac{2 + \sqrt{7}}{3} \approx \dfrac{2 + 2.64}{3} = 1.54$.

(iv) The two conditions to check are $s_2 + s_3 > s_1$ (always needed) and $s_1 > s_3$ or $s_3 > s_1$.

(v) Check $s_2 + s_3 > s_1$:

$$(a^2 + a - 2) + (2a^2 + a - 1) > 5a^2 - 3a - 2$$

$$3a^2 + 2a - 3 > 5a^2 - 3a - 2$$

$$0 > 2a^2 - 5a + 1$$

(vi) Solve $2a^2 - 5a + 1 < 0$. The roots of $2a^2 - 5a + 1 = 0$ are:

$$a = \frac{5 \pm \sqrt{25-8}}{4} = \frac{5 \pm \sqrt{17}}{4}$$

Let $a_1 = \frac{5-\sqrt{17}}{4} \approx \frac{5-4.12}{4} \approx 0.22$. Let $a_2 = \frac{5+\sqrt{17}}{4} \approx \frac{5+4.12}{4} \approx 2.28$. For $2a^2 - 5a + 1 < 0$, we need $a \in (a_1, a_2)$, so $\mathbf{a} \in \left(\frac{\mathbf{5-\sqrt{17}}}{\mathbf{4}}, \frac{\mathbf{5+\sqrt{17}}}{\mathbf{4}}\right)$.

(vii) The final set is the intersection of $a \in (1, \infty)$ and $a \in \left(\frac{5-\sqrt{17}}{4}, \frac{5+\sqrt{17}}{4}\right)$. Since $\frac{5-\sqrt{17}}{4} \approx 0.22$ and $1 > 0.22$, the intersection is:

$$\mathbf{a} \in \left(\mathbf{1}, \frac{\mathbf{5+\sqrt{17}}}{\mathbf{4}}\right)$$

Answer: $\left(1, \frac{5+\sqrt{17}}{4}\right)$

15. **Question:** If $x \in R$ prove that the maximum value of $2(a-x)(x+\sqrt{x^2+b^2})$ is a^2+b^2

Solution:

(i) Let $f(x) = 2(a-x)(x+\sqrt{x^2+b^2})$.

(ii) The expression suggests the substitution $x = b\tan\theta$. This leads to $\sqrt{x^2+b^2} = b\sec\theta$.

$$f(\theta) = 2(a - b\tan\theta)(b\tan\theta + b\sec\theta)$$
$$f(\theta) = 2b(a - b\tan\theta)(\tan\theta + \sec\theta)$$
$$f(\theta) = 2b\left(a - b\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\sin\theta+1}{\cos\theta}\right)$$
$$f(\theta) = \frac{2b}{\cos^2\theta}(a\cos\theta - b\sin\theta)(1+\sin\theta)$$

This is complicated.

(iii) **Alternative method: Use the derivative to find the critical points.** Let $g(x) = x + \sqrt{x^2+b^2}$. $f(x) = 2(a-x)g(x)$.

$$g'(x) = 1 + \frac{2x}{2\sqrt{x^2+b^2}} = 1 + \frac{x}{\sqrt{x^2+b^2}} = \frac{\sqrt{x^2+b^2}+x}{\sqrt{x^2+b^2}} = \frac{g(x)}{\sqrt{x^2+b^2}}$$
$$f'(x) = 2\left[(-1)g(x) + (a-x)g'(x)\right]$$
$$f'(x) = 2\left[-g(x) + (a-x)\frac{g(x)}{\sqrt{x^2+b^2}}\right]$$
$$f'(x) = 2g(x)\left[-1 + \frac{a-x}{\sqrt{x^2+b^2}}\right]$$

(iv) For critical points, set $f'(x) = 0$. Since $g(x) = x + \sqrt{x^2+b^2} > x + \sqrt{x^2} = x + |x| \geq 0$, we must have:

$$-1 + \frac{a-x}{\sqrt{x^2+b^2}} = 0 \implies \sqrt{x^2+b^2} = a - x$$

(v) The condition $\sqrt{x^2+b^2} = a - x$ requires $a - x \geq 0$, or $\mathbf{x \leq a}$. Square both sides:

$$x^2 + b^2 = (a-x)^2 = a^2 - 2ax + x^2$$
$$b^2 = a^2 - 2ax$$
$$2ax = a^2 - b^2$$

(vi) **Case:** $a \neq 0$

$$x_0 = \frac{a^2-b^2}{2a}$$

Check $x_0 \leq a$:

$$a - x_0 = a - \frac{a^2-b^2}{2a} = \frac{2a^2-a^2+b^2}{2a} = \frac{a^2+b^2}{2a}$$

Since $b^2 > 0$, $a^2 + b^2 > 0$. If $a > 0$, then $a - x_0 > 0$, so $x_0 < a$. If $a < 0$, then $a - x_0 < 0$, so $x_0 > a$. But we need $a - x_0 \geq 0$. So, we must have $\mathbf{a > 0}$.

(vii) **Calculate $f(x_0)$:** At $x = x_0$, $x_0 + \sqrt{x_0^2 + b^2} = x_0 + (a - x_0) = a$.

$$f(x_0) = 2(a - x_0)(x_0 + \sqrt{x_0^2 + b^2})$$
$$f(x_0) = 2(a - x_0)(a)$$

Substitute $a - x_0 = \dfrac{a^2 + b^2}{2a}$:

$$f(x_0) = 2a\left(\frac{a^2 + b^2}{2a}\right) = \mathbf{a^2 + b^2}$$

(viii) Since $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, this critical point is the global maximum.

(ix) **Case:** $a = 0$ $f(x) = 2(-x)(x + \sqrt{x^2 + b^2})$. $f'(x) = 2g(x)[-1 + \dfrac{-x}{\sqrt{x^2 + b^2}}] < 0$ (since $-x \le \sqrt{x^2 + b^2}$ for $b \neq 0$). The function is strictly decreasing and has no critical point, so the maximum is not b^2. The limit as $x \to -\infty$ is 0. However, the problem implies $a^2 + b^2$ is the maximum, which suggests $a > 0$ is assumed.

16. **Question:** For what values of the parameter a, the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$ has at least two distinct negative roots.

Solution:

(i) The equation is a reciprocal equation: $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$. Since $x = 0$ is not a root, divide by x^2:

$$x^2 + 2ax + 1 + \frac{2a}{x} + \frac{1}{x^2} = 0$$
$$\left(x^2 + \frac{1}{x^2}\right) + 2a\left(x + \frac{1}{x}\right) + 1 = 0$$

(ii) Let $y = x + \dfrac{1}{x}$. Then $x^2 + \dfrac{1}{x^2} = y^2 - 2$.

$$(y^2 - 2) + 2ay + 1 = 0$$
$$\mathbf{y^2 + 2ay - 1 = 0}$$

(iii) The condition is that the original equation has at least two distinct negative roots. The relationship between x and y is given by $x^2 - yx + 1 = 0$.

- For x to be a real root, the discriminant of $x^2 - yx + 1 = 0$ must be ≥ 0:
$$D_x = y^2 - 4 \ge 0 \implies y^2 \ge 4 \implies |\mathbf{y}| \ge \mathbf{2}$$
- For x to be a negative root, $x < 0$. Since $x^2 - yx + 1 = 0$ has product of roots $1 > 0$, the two roots (if real) must have the same sign. The sum of the roots is y. For the roots to be negative, the sum must be negative: $\mathbf{y < 0}$.

(iv) Combining the conditions on y: $\mathbf{y \le -2}$.

(v) Let $g(y) = y^2 + 2ay - 1$. For the original equation to have at least two distinct negative roots, $g(y) = 0$ must have at least one root y_0 such that $y_0 < -2$.

(vi) The roots of $g(y) = 0$ are $y = \dfrac{-2a \pm \sqrt{4a^2 + 4}}{2} = -a \pm \sqrt{a^2 + 1}$. Let $y_1 = -a - \sqrt{a^2 + 1}$ and $y_2 = -a + \sqrt{a^2 + 1}$. Since $\sqrt{a^2 + 1} > 0$, we have $y_1 < y_2$.

(vii) For $y_1 < -2$, we require:

$$-a - \sqrt{a^2 + 1} < -2$$
$$2 - a < \sqrt{a^2 + 1}$$

(viii) **Case 1:** $2 - a < 0$ ($\mathbf{a > 2}$) Since $\sqrt{a^2 + 1} > 0$, the inequality is always true.

(ix) **Case 2:** $2 - a \ge 0$ ($\mathbf{a \le 2}$) Square both sides:

$$(2 - a)^2 < a^2 + 1$$
$$4 - 4a + a^2 < a^2 + 1$$
$$4 - 4a < 1 \implies 3 < 4a \implies \mathbf{a > 3/4}$$

(x) Combining Case 1 and Case 2, the condition for $y_1 < -2$ is $\mathbf{a > 3/4}$.

(xi) The range of y_1 is $(-\infty, -2)$ for $a > 3/4$.

(xii) The second root $y_2 = -a + \sqrt{a^2 + 1}$. Since $\sqrt{a^2 + 1} > a$ for all real a, $y_2 > 0$. Since $y_2 > 0$, y_2 cannot lead to negative roots (since $y < 0$ is required).

(xiii) The four original roots are generated from y_1 (two distinct negative roots) and y_2 (two distinct positive roots).

(xiv) For the original equation to have at least two distinct negative roots, we need $y_1 < -2$, which is $\mathbf{a > 3/4}$.

Answer: $a \in (3/4, \infty)$