SOLUTIONS TO PROBLEMS FROM JEE ADVANCED PREVIOUS YEARS

Complex Numbers

SET 1

Fill in the Blanks

1. If the expression $\frac{[\sin(\frac{x}{2})+\cos(\frac{x}{2})-i\tan(x)]}{[1+2i\sin(\frac{x}{2})]}$ is real, then the set of all possible values of x is ....

Solution: Let the given expression be Z.

$$Z = \frac{\left[\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) - i\tan(x)\right]}{\left[1 + 2i\sin\left(\frac{x}{2}\right)\right]}$$

For Z to be real, $Z = \overline{Z}$, which means the imaginary part of Z must be zero. We multiply the numerator and denominator by the conjugate of the denominator:

$$\begin{aligned}
Z &= \frac{\left[\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) - i\tan(x)\right]\left[1 - 2i\sin\left(\frac{x}{2}\right)\right]}{\left[1 + 2i\sin\left(\frac{x}{2}\right)\right]\left[1 - 2i\sin\left(\frac{x}{2}\right)\right]} \\
&= \frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - 2i\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - i\tan x + 2i^2\tan x\sin\frac{x}{2}}{1^2 + \left(2\sin\frac{x}{2}\right)^2} \\
&= \frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - 2\tan x\sin\frac{x}{2} + i\left[-2\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - \tan x\right]}{1 + 4\sin^2\frac{x}{2}}
\end{aligned}$$

For Z to be real, the imaginary part must be zero:

$$-2\sin\left(\frac{x}{2}\right)\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right) - \tan(x) = 0$$

$$-2\sin^2\left(\frac{x}{2}\right) - 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) - \tan(x) = 0$$

Using the double angle identity, $2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x$:

$$-(1 - \cos x) - \sin x - \tan x = 0$$

$$\cos x + \sin x + \tan x = 1$$

Since $\tan x$ must be defined, $x \neq (m+\frac{1}{2})\pi$, $m \in \mathbb{Z}$. Let's check the case where $\sin x = 0$. This implies $x = n\pi$, $n \in \mathbb{Z}$. If $x = 2k\pi$ (even multiple of π): $\cos(2k\pi) + \sin(2k\pi) + \tan(2k\pi) = 1 + 0 + 0 = 1$. This is true. If $x = (2k+1)\pi$ (odd multiple of π): $\cos((2k+1)\pi) + \sin((2k+1)\pi) + \tan((2k+1)\pi) = -1 + 0 + 0 = -1 \neq 1$. This is false.

So, $x = 2n\pi$, $n \in \mathbb{Z}$.

Answer: $x = 2n\pi, n \in \mathbb{Z}$

2. For any two complex numbers z_1, z_2 and any real numbers a and b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 =$ .....

Solution: We use the property $|z|^2 = z\overline{z}$.

$$\begin{aligned}
|az_1 - bz_2|^2 &= (az_1 - bz_2)(\overline{az_1 - bz_2}) \\
&= (az_1 - bz_2)(a\overline{z_1} - b\overline{z_2}) \quad (\text{since } a, b \in \mathbb{R}) \\
&= a^2 z_1\overline{z_1} - abz_1\overline{z_2} - abz_2\overline{z_1} + b^2 z_2\overline{z_2} \\
&= a^2|z_1|^2 - ab(z_1\overline{z_2} + \overline{z_1}z_2) + b^2|z_2|^2
\end{aligned}$$

$$\begin{aligned}
|bz_1 + az_2|^2 &= (bz_1 + az_2)(\overline{bz_1 + az_2}) \\
&= (bz_1 + az_2)(b\overline{z_1} + a\overline{z_2}) \\
&= b^2 z_1\overline{z_1} + abz_1\overline{z_2} + abz_2\overline{z_1} + a^2 z_2\overline{z_2} \\
&= b^2|z_1|^2 + ab(z_1\overline{z_2} + \overline{z_1}z_2) + a^2|z_2|^2
\end{aligned}$$

Adding the two expressions:

$$\begin{aligned}|az_1 - bz_2|^2 + |bz_1 + az_2|^2 &= (a^2|z_1|^2 + b^2|z_2|^2) + (b^2|z_1|^2 + a^2|z_2|^2) \\ &= (a^2 + b^2)|z_1|^2 + (a^2 + b^2)|z_2|^2 \\ &= (a^2 + b^2)(|z_1|^2 + |z_2|^2)\end{aligned}$$

This is a generalization of the Parallelogram Law.

Answer: $(a^2 + b^2)(|z_1|^2 + |z_2|^2)$

3. If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i, z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle then a = ... and b = ...

Solution: For an equilateral triangle with vertices z_1, z_2, z_3, the side lengths must be equal, i.e., $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$.

1. $|z_1 - z_3| = |z_2 - z_3| \implies |z_1| = |z_2|$

$$|a + i| = |1 + bi|$$

$$\sqrt{a^2 + 1^2} = \sqrt{1^2 + b^2}$$

$$a^2 + 1 = 1 + b^2 \implies a^2 = b^2$$

Since $a, b \in (0, 1)$, we must have $\mathbf{a} = \mathbf{b}$.

2. The condition for z_1, z_2, z_3 to form an equilateral triangle is $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$. With $z_3 = 0$, this simplifies to $z_1^2 + z_2^2 = z_1z_2$. Substituting $z_1 = a + i$ and $z_2 = 1 + ai$ (since $a = b$):

$$(a + i)^2 + (1 + ai)^2 = (a + i)(1 + ai)$$

$$(a^2 - 1 + 2ai) + (1 - a^2 + 2ai) = (a - a) + i(a^2 + 1)$$

$$4ai = i(a^2 + 1)$$

Since $a \in (0, 1)$, $i \neq 0$, so we divide by i:

$$4a = a^2 + 1$$

$$a^2 - 4a + 1 = 0$$

Using the quadratic formula for a:

$$a = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \frac{4 \pm \sqrt{16 - 4}}{2} = \frac{4 \pm \sqrt{12}}{2}$$

$$a = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

We are given that $a \in (0, 1)$. $2 + \sqrt{3} \approx 2 + 1.732 = 3.732$ (rejected). $2 - \sqrt{3} \approx 2 - 1.732 = 0.268$. Since $0 < 0.268 < 1$, this is accepted.

Therefore, $a = 2 - \sqrt{3}$. Since $a = b$, we have $b = 2 - \sqrt{3}$.

Answer: $a = 2 - \sqrt{3}$ and $b = 2 - \sqrt{3}$

TRUE / FALSE

4. For complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$, if $x_1 \leq x_2$ and $y_1 \leq y_2$. Then for all complex numbers z with $1 \cap z$, we have $\frac{1-z}{1+z} \cap 0$

Solution: The relation $z_1 \cap z_2$ is defined as $x_1 \leq x_2$ and $y_1 \leq y_2$. The condition $1 \cap z$ means: $1 \cap (x + iy)$, so $1 \leq x$ and $0 \leq y$.

We need to check if $\frac{1-z}{1+z} \cap 0$. Let $w = \frac{1-z}{1+z} = u + iv$. The condition $w \cap 0$ means $u \leq 0$ and $v \leq 0$.

$$w = \frac{1 - (x + iy)}{1 + (x + iy)} = \frac{(1 - x) - iy}{(1 + x) + iy}$$

Multiplying by the conjugate of the denominator:

$$w = \frac{((1-x)-iy)((1+x)-iy)}{((1+x)+iy)((1+x)-iy)} = \frac{(1-x)(1+x) - i(1-x)y - i(1+x)y - y^2}{(1+x)^2+y^2}$$

$$w = \frac{(1-x^2-y^2) - i(2y)}{(1+x)^2+y^2}$$

The real part is $u = \frac{1-x^2-y^2}{(1+x)^2+y^2}$. The imaginary part is $v = \frac{-2y}{(1+x)^2+y^2}$.

From $1 \cap z$, we have $x \geq 1$ and $y \geq 0$.

1. Check $v \leq 0$: Since $y \geq 0$ and the denominator $(1+x)^2+y^2$ is positive (assuming $z \neq -1$), we have $v = \frac{-2y}{(1+x)^2+y^2} \leq 0$. This is true.

2. Check $u \leq 0$: Since $x \geq 1$, we have $x^2 \geq 1$. Thus, $1-x^2 \leq 0$. Since $y \geq 0$, $y^2 \geq 0$. Therefore, $1-x^2-y^2 \leq 0$. Since the numerator is non-positive and the denominator is positive, we have $u = \frac{1-x^2-y^2}{(1+x)^2+y^2} \leq 0$. This is true.

Since both $u \leq 0$ and $v \leq 0$ are true, $\frac{1-z}{1+z} \cap 0$ is true.

Answer: TRUE

5. If the complex numbers, z_1, z_2 and z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 = 0$

   Solution: The condition $|z_1| = |z_2| = |z_3|$ means the vertices of the triangle lie on a circle centered at the origin, i.e., the origin $O(0)$ is the circumcenter of the triangle $\triangle z_1z_2z_3$.

   For an equilateral triangle, the circumcenter coincides with the centroid. The centroid of $\triangle z_1z_2z_3$ is $G = \frac{z_1+z_2+z_3}{3}$. Since the circumcenter is $O(0)$, we have $G = 0$.

$$\frac{z_1+z_2+z_3}{3} = 0$$

$$z_1 + z_2 + z_3 = 0$$

   The statement is true.

   Answer: TRUE

OBJECTIVE TYPE Only one option is correct

6. The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$, is

   A. 8 B. 16 C. 12 D. none of these

   Solution: First, simplify the complex number inside the parenthesis:

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{(1+i)^2}{1^2-i^2} = \frac{1+2i+i^2}{1-(-1)} = \frac{1+2i-1}{2} = \frac{2i}{2} = i$$

   The equation becomes $i^n = 1$. The powers of i repeat in a cycle of 4: $i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$. The smallest positive integer n for which $i^n = 1$ is $\mathbf{n = 4}$.

   Answer: The correct option is none of these (as the answer is 4).

7. The complex numbers z=x+iy which satisfy the equation $\left|\frac{z-5i}{z+5i}\right| = 1$ lie on :

   A. the x axis B. the straight line y = 5 C. a circle passing through the origin D. none of these

   Solution: The given equation is $\left|\frac{z-5i}{z+5i}\right| = 1$. This can be written as $|z-5i| = |z-(-5i)|$. This equation states that the distance of z from the point $z_1 = 5i$ is equal to the distance of z from the point $z_2 = -5i$. The locus of a point equidistant from two fixed points is the **perpendicular bisector** of the segment joining the two points. The points $5i$ and $-5i$ lie on the imaginary axis. The midpoint of the segment joining $5i$ and $-5i$ is $\frac{5i+(-5i)}{2} = 0$. The perpendicular bisector of the imaginary axis segment passing through the origin is the **real axis** (or x-axis).

Alternatively, using coordinates $z = x + iy$:

$$|x + iy - 5i| = |x + iy + 5i|$$
$$|x + i(y - 5)| = |x + i(y + 5)|$$
$$\sqrt{x^2 + (y-5)^2} = \sqrt{x^2 + (y+5)^2}$$

Squaring both sides:

$$x^2 + y^2 - 10y + 25 = x^2 + y^2 + 10y + 25$$
$$-10y = 10y$$
$$20y = 0 \implies y = 0$$

Since $z = x + iy = x$, z lies on the real axis (x-axis).

Answer: the x axis

8. If z=$(\frac{\sqrt{3}}{2} + \frac{i}{2})^5 + (\frac{\sqrt{3}}{2} - \frac{i}{2})^5$, then :
A. Re(z)=0 B. Im(z) =0 C. $Re(z) > 0, Im(z) > 0$ D. $Re(z) > 0, Im(z) < 0$

Solution: We can write the terms in polar form:

$$\frac{\sqrt{3}}{2} + \frac{i}{2} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = e^{i\frac{\pi}{6}}$$
$$\frac{\sqrt{3}}{2} - \frac{i}{2} = \cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right) = e^{-i\frac{\pi}{6}}$$

The expression for z becomes:

$$\begin{aligned} z &= \left(e^{i\frac{\pi}{6}}\right)^5 + \left(e^{-i\frac{\pi}{6}}\right)^5 \\ &= e^{i\frac{5\pi}{6}} + e^{-i\frac{5\pi}{6}} \end{aligned}$$

Using Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$:

$$\begin{aligned} z &= \left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) + \left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right) \\ &= \left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) + \left(\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right)\right) \\ &= 2\cos\left(\frac{5\pi}{6}\right) \end{aligned}$$

Since $\cos(\frac{5\pi}{6}) = \cos(\pi - \frac{\pi}{6}) = -\cos(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$.

$$z = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$$

Since $z = -\sqrt{3}$, we have $\text{Im}(z) = 0$.

Answer: Im(z) =0

9. The inequality $|z - 4| < |z - 2|$ represents the region given by :
A. $Re(z) \geq 0$ B. $Re(z) < 0$ C. $Re(z) > 0$ D. none of these

Solution: The inequality $|z - 4| < |z - 2|$ states that the distance of the point z from $z_1 = 4$ is strictly less than the distance of z from $z_2 = 2$. The boundary $|z - 4| = |z - 2|$ is the perpendicular bisector of the segment connecting $z_1 = 4$ and $z_2 = 2$ on the real axis. Midpoint: $\frac{4+2}{2} = 3$. The perpendicular bisector is the line $\text{Re}(z) = 3$ (or $x = 3$).

The region $|z - 4| < |z - 2|$ is the side of the line $x = 3$ that contains the point $z_2 = 2$. Since $2 - 4 = -2$ and $2 - 2 = 0$, and $|-2| < |0|$ is false, we must check a point. Test a point, say $z = 0$: $|0 - 4| < |0 - 2| \implies 4 < 2$, which is false. Test a point, say $z = 1$: $|1 - 4| < |1 - 2| \implies 3 < 1$, which is false. Test a point, say $z = 3.5$: $|3.5 - 4| < |3.5 - 2| \implies 0.5 < 1.5$, which is true.

Since $z = 3.5$ lies in the region, and 3.5 is to the right of 3, the region is $Re(z) > 3$.

Let's use the algebraic method with $z = x + iy$:

$$|x + iy - 4| < |x + iy - 2|$$

$$\sqrt{(x-4)^2 + y^2} < \sqrt{(x-2)^2 + y^2}$$

Squaring both sides (since both sides are non-negative):

$$(x-4)^2 + y^2 < (x-2)^2 + y^2$$

$$x^2 - 8x + 16 < x^2 - 4x + 4$$

$$-8x + 16 < -4x + 4$$

$$12 < 4x$$

$$3 < x$$

The region is $\mathrm{Re}(z) > 3$.

Comparing with the options, $\mathrm{Re}(z) > 3$ is a subset of $\mathrm{Re}(z) > 0$. However, the options given are incorrect for the JEE standard. Assuming there is a typo in the options and one should be $\mathrm{Re}(z) > 3$ or similar, we choose the closest general condition that is true for $x > 3$. Since $x > 3$ is the exact answer, and $x > 3 \implies x > 0$, $\mathrm{Re}(z) > 0$ is a plausible choice if the option $\mathrm{Re}(z) > 3$ is missing, but $\mathrm{Re}(z) > 0$ is too broad. None of the options correctly describe the region $x > 3$. If the question or options intended to be simpler, let's re-examine.

Based on the typical context of this question type, the intended answer is often the line dividing the points, which is $x = 3$. The region is $x > 3$. Since $x > 3$ is not an option, and $x > 0$ is too general, the mathematically correct option from the given choices is "none of these".

Answer: none of these

10. If z = x+iy and w = $\frac{(1-iz)}{(z-i)}$ then $|w| = 1$ implies that, in the complex plane :
A. z lies on the imaginary axis B. z lies on the real axis C. z lies on the unit circle D. none of these

Solution: Given $|w| = 1$, we have:

$$\left|\frac{1-iz}{z-i}\right| = 1$$

$$|1 - iz| = |z - i|$$

We use the property $|iz| = |i||z| = 1 \cdot |z| = |z|$.

$$|1 - iz| = |-(iz - 1)| = |-(i(z - \frac{1}{i}))| = |-(i(z + i))|$$

$$|1 - iz| = |-i(z + i)| = |-i||z + i| = 1 \cdot |z + i| = |z + i|$$

So the equation becomes:

$$|z + i| = |z - i|$$

This states that the distance of the point z from $z_1 = -i$ is equal to the distance of z from $z_2 = i$. The locus of z is the perpendicular bisector of the segment joining i and $-i$. The points i and $-i$ are on the imaginary axis. The perpendicular bisector is the **real axis** (x-axis).

Alternatively, using coordinates $z = x + iy$:

$$|z + i| = |z - i|$$
$$|x + iy + i| = |x + iy - i|$$
$$|x + i(y + 1)| = |x + i(y - 1)|$$
$$\sqrt{x^2 + (y+1)^2} = \sqrt{x^2 + (y-1)^2}$$

Squaring both sides:

$$x^2 + y^2 + 2y + 1 = x^2 + y^2 - 2y + 1$$

$$2y = -2y$$

$$4y = 0 \implies y = 0$$

Since $z = x + iy = x$, z lies on the real axis.

Answer: z lies on the real axis

11. The point z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order , if and only if :
A. $z_1 + z_4 = z_2 + z_3$ B. $z_1 + z_3 = z_2 + z_4$ C. $z_1 + z_2 = z_3 + z_4$ D. none of these

Solution: In a parallelogram, the diagonals bisect each other. Let the vertices be $A(z_1), B(z_2), C(z_3), D(z_4)$ taken in order. The diagonals are AC and BD. The midpoint of AC is $\frac{z_1+z_3}{2}$. The midpoint of BD is $\frac{z_2+z_4}{2}$.

For the diagonals to bisect each other, the midpoints must be the same:

$$\frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$$

$$z_1 + z_3 = z_2 + z_4$$

Answer: $z_1 + z_3 = z_2 + z_4$

12. If a,b,c and u,v,w are complex numbers representing the vertices of two triangle such that c=(1-r)a+rb and w=(1-r)u+rv, where r is a complex number, then the two triangles :
A. have the same area B. are similar C. are congruent D. none of these

Solution: The given relations are:

$$c = (1 - r)a + rb \implies c - a = r(b - a)$$

$$w = (1 - r)u + rv \implies w - u = r(v - u)$$

From the first relation, $\frac{c-a}{b-a} = r$. From the second relation, $\frac{w-u}{v-u} = r$.
Equating the two expressions for r:

$$\frac{c - a}{b - a} = \frac{w - u}{v - u}$$

This can be rewritten as:

$$\frac{c - a}{w - u} = \frac{b - a}{v - u}$$

This equality of ratios means the ratio of corresponding side lengths $|c-a|/|w-u|$ and $|b-a|/|v-u|$ are equal, and the angle between the corresponding sides are equal:

$$\left|\frac{c - a}{w - u}\right| = \left|\frac{b - a}{v - u}\right| \implies \frac{|c - a|}{|w - u|} = \frac{|b - a|}{|v - u|}$$

$$\arg\left(\frac{c - a}{w - u}\right) = \arg\left(\frac{b - a}{v - u}\right) \implies \arg(c - a) - \arg(w - u) = \arg(b - a) - \arg(v - u)$$

However, the first form $\frac{c-a}{b-a} = \frac{w-u}{v-u}$ shows that the mapping that transforms the vertices A, B, C to U, V, W is an angle-preserving transformation that scales lengths equally (since the complex number r is the same). More precisely, the ratio of the complex numbers representing sides $\vec{AC}$ to $\vec{AB}$ is equal to the ratio of complex numbers representing sides $\vec{UW}$ to $\vec{UV}$.

$$\frac{z_C - z_A}{z_B - z_A} = \frac{z_W - z_U}{z_V - z_U}$$

This is the condition for the two triangles ABC and UVW to be **similar** in the same sense (i.e., with the same orientation).

Answer: are similar

13. The value of $\sum_{k=1}^{6}(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7})$ is :
A. -1 B. 0 C. -i D. i

Solution: Let the sum be S.

$$S = \sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$$

We can rewrite the term inside the summation:

$$\begin{aligned}\sin\theta - i\cos\theta &= \sin\theta - i\cos\theta - i^2\sin\theta + i^2\sin\theta \quad \text{(redundant step)}\\ &= -i(\cos\theta + i\sin\theta)\\ &= -ie^{i\theta}\end{aligned}$$

Here, $\theta = \frac{2\pi k}{7}$. Let $\omega = e^{i\frac{2\pi}{7}}$. The term is $-ie^{i\frac{2\pi k}{7}} = -i(\omega)^k$.

$$S = \sum_{k=1}^{6}(-i\omega^k) = -i\sum_{k=1}^{6}\omega^k$$

$\omega = e^{i\frac{2\pi}{7}}$ is a 7-th root of unity ($\omega^7 = 1$). For n-th roots of unity, the sum of the roots is zero: $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$. Here $n = 7$, so $1 + \omega + \omega^2 + \cdots + \omega^6 = 0$. Therefore, $\sum_{k=1}^{6}\omega^k = \omega + \omega^2 + \cdots + \omega^6 = -1$.

$$S = -i(-1) = i$$

Answer: i

14. If z_1 and z_2 are two non zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg z_1 - \arg z_2$ is equal to :
A. $-\pi$ B. $-\frac{\pi}{2}$ C. 0 D. $\frac{\pi}{2}$ E. π

Solution: The triangle inequality states that $|z_1 + z_2| \leq |z_1| + |z_2|$. Equality holds, i.e., $|z_1 + z_2| = |z_1| + |z_2|$, if and only if z_1 and z_2 lie on the same ray from the origin. This is equivalent to saying that $z_1 = \lambda z_2$ for some real number $\lambda > 0$.

Since $z_1 = \lambda z_2$ with $\lambda > 0$, the arguments must be the same:

$$\arg(z_1) = \arg(\lambda z_2)$$

Since $\lambda > 0$, $\arg(\lambda) = 0$.

$$\arg(z_1) = \arg(\lambda) + \arg(z_2) + 2k\pi \quad (k \in \mathbb{Z})$$

$$\arg(z_1) = 0 + \arg(z_2) + 2k\pi$$

$$\arg z_1 - \arg z_2 = 2k\pi$$

If we restrict the principal argument to $\arg z \in (-\pi, \pi]$, then $2k\pi = 0$.

The difference $\arg z_1 - \arg z_2$ must be 0 (or a multiple of 2π).

Answer: 0

15. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other , for :
A. x $=n\pi$ B. x=0 C. x=$(n+\frac{1}{2})\pi$ D. no value of x

Solution: Let $z_1 = \sin x + i\cos 2x$ and $z_2 = \cos x - i\sin 2x$. z_1 and z_2 are conjugate to each other if $z_1 = \overline{z_2}$.

$$\overline{z_2} = \overline{\cos x - i\sin 2x} = \cos x + i\sin 2x$$

The condition $z_1 = \overline{z_2}$ becomes:

$$\sin x + i\cos 2x = \cos x + i\sin 2x$$

Equating the real and imaginary parts: 1. Real part: $\sin x = \cos x$

$$\tan x = 1 \implies x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

2. Imaginary part: $\cos 2x = \sin 2x$

$$\tan 2x = 1 \implies 2x = m\pi + \frac{\pi}{4}, m \in \mathbb{Z}$$

$$x = \frac{m\pi}{2} + \frac{\pi}{8}, m \in \mathbb{Z}$$

For both conditions to be satisfied simultaneously, we need the intersection of the two solution sets:

$$n\pi + \frac{\pi}{4} = \frac{m\pi}{2} + \frac{\pi}{8}$$

Multiplying by 8:

$$8n\pi + 2\pi = 4m\pi + \pi$$
$$8n\pi - 4m\pi = -\pi$$
$$(8n - 4m)\pi = -\pi$$
$$4(2n - m) = -1$$

Since n and m are integers, $2n - m$ is an integer. Let $k = 2n - m$.

$$4k = -1 \implies k = -\frac{1}{4}$$

Since k must be an integer, there are no integer values for n and m that satisfy this equation.

Therefore, there is no value of x for which the two complex numbers are conjugates.

Answer: no value of x

16. If $\omega \neq 1$ is a cube root of unity and $(1+\omega)^7 = A + B\omega$, then A and B are respectively :
A. 0,1 B. 1,1 C. $\overline{\omega}$ D. $-\overline{\omega}$

Solution: Since ω is a cube root of unity and $\omega \neq 1$, we have:

$$1 + \omega + \omega^2 = 0 \implies 1 + \omega = -\omega^2$$

Also, $\omega^3 = 1$.

Substitute $1 + \omega = -\omega^2$ into the given expression:

$$\begin{aligned}(1+\omega)^7 &= (-\omega^2)^7 \\ &= -(\omega^2)^7 \\ &= -\omega^{14} \\ &= -(\omega^{12} \cdot \omega^2) \\ &= -(((\omega^3)^4) \cdot \omega^2) \\ &= -(1^4 \cdot \omega^2) \\ &= -\omega^2\end{aligned}$$

We need to express $-\omega^2$ in the form $A + B\omega$. Since $1 + \omega + \omega^2 = 0$, we have $-\omega^2 = 1 + \omega$.

So, $(1+\omega)^7 = 1 + \omega$. Comparing this with $A + B\omega$, we get:

$$A + B\omega = 1 + 1 \cdot \omega$$

Since $1, \omega, \omega^2$ are linearly independent over $\mathbb{R}$, and A, B are typically real (or A, B are the unique coefficients in the basis $\{1, \omega\}$), we equate the coefficients.

$$A = 1 \quad \text{and} \quad B = 1$$

Answer: 1,1