Detailed Solutions: Limits and Trigonometry – Set 3

Multiple Choice Questions Solutions

1. **Question:** Show that the following limit equals e:

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}$$

Solution: This is the fundamental definition of the mathematical constant e. The limit is of the indeterminate form 1^∞.

$$L=\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}=e$$

Answer: e (Option b).

2. **Question:** Evaluate the following limit:

$$\lim_{x\to 1}\frac{x^9-1}{x^{14}-1}$$

Solution: This limit is of the form $\frac{0}{0}$. We use the standard formula for algebraic limits, $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$:

$$L=\lim_{x\to 1}\frac{\frac{x^9-1}{x-1}}{\frac{x^{14}-1}{x-1}}=\frac{\lim_{x\to 1}\frac{x^9-1^9}{x-1}}{\lim_{x\to 1}\frac{x^{14}-1^{14}}{x-1}}$$

$$L=\frac{9\cdot 1^{9-1}}{14\cdot 1^{14-1}}=\frac{9}{14}$$

Alternatively, using L'Hôpital's Rule:

$$L=\lim_{x\to 1}\frac{9x^8}{14x^{13}}=\frac{9(1)^8}{14(1)^{13}}=\frac{9}{14}$$

Answer: $\frac{9}{14}$ (Option a).

3. **Question:** Evaluate the following limit:

$$\lim_{x\to\tan^{-1}(3)}\frac{\tan^2 x-2\tan x-3}{\tan^2 x-4\tan x+3}$$

Solution: Let $y=\tan x$. As $x\to\tan^{-1}(3)$, $y\to 3$. The limit becomes:

$$L=\lim_{y\to 3}\frac{y^2-2y-3}{y^2-4y+3}$$

Factor the quadratic expressions:

$$L=\lim_{y\to 3}\frac{(y-3)(y+1)}{(y-3)(y-1)}$$

Since $y\neq 3$ as $y\to 3$, we can cancel $(y-3)$:

$$L=\lim_{y\to 3}\frac{y+1}{y-1}=\frac{3+1}{3-1}=\frac{4}{2}=2$$

Answer: 2 (Option c).

4. **Question:** Evaluate the limit:

$$\lim_{x\to 2}\frac{3^{\frac{x}{2}}-3}{3^x-9}$$

Solution: The limit is of the form $\frac{0}{0}$. We can recognize that the denominator is a difference of squares involving the term in the numerator: $3^x - 9 = (3^{x/2})^2 - 3^2$.

$$L = \lim_{x \to 2} \frac{3^{\frac{x}{2}} - 3}{(3^{\frac{x}{2}} - 3)(3^{\frac{x}{2}} + 3)}$$

Since $x \neq 2$, $3^{x/2} - 3 \neq 0$, so we can cancel the term:

$$L = \lim_{x \to 2} \frac{1}{3^{\frac{x}{2}} + 3}$$

Substituting $x = 2$:

$$L = \frac{1}{3^{\frac{2}{2}} + 3} = \frac{1}{3 + 3} = \frac{1}{6}$$

Answer: $\frac{1}{6}$ (Note: Since $\frac{1}{6}$ is not an option, there may be a typo in the provided options. $\frac{1}{6}$ is the mathematically correct answer).

5. **Question:** Evaluate the limit:

$$\lim_{\alpha \to 0} \frac{\sin^2 \alpha - \sin^2 \beta}{\alpha^2 - \beta^2}$$

Solution: For the limit to result in one of the options involving β, it is highly likely the intended limit was $\alpha \to \beta$, not $\alpha \to 0$. We will solve for the likely intended question, $\lim_{\alpha \to \beta}$.

Using the difference of squares identity $\sin^2 A - \sin^2 B = \sin(A - B)\sin(A + B)$:

$$L = \lim_{\alpha \to \beta} \frac{\sin(\alpha - \beta)\sin(\alpha + \beta)}{(\alpha - \beta)(\alpha + \beta)}$$

We group the terms to use the fundamental trigonometric limit $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$:

$$L = \lim_{\alpha \to \beta} \left(\frac{\sin(\alpha - \beta)}{\alpha - \beta} \right) \cdot \left(\frac{\sin(\alpha + \beta)}{\alpha + \beta} \right)$$

As $\alpha \to \beta$, the first term $\to 1$, and the second term approaches $\frac{\sin(\beta + \beta)}{\beta + \beta}$:

$$L = 1 \cdot \frac{\sin(2\beta)}{2\beta}$$

Answer: $\frac{\sin 2\beta}{2\beta}$ (Option d).

6. **Question:** Evaluate the limit:

$$\lim_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2}\sin x}$$

Solution: The limit is of the form $\frac{0}{0}$. We use L'Hôpital's Rule:

$$L = \lim_{x \to \frac{\pi}{4}} \frac{\frac{d}{dx}(1 - \tan x)}{\frac{d}{dx}(1 - \sqrt{2}\sin x)} = \lim_{x \to \frac{\pi}{4}} \frac{-\sec^2 x}{-\sqrt{2}\cos x}$$

Substitute $x = \frac{\pi}{4}$:

$$L = \frac{\sec^2(\frac{\pi}{4})}{\sqrt{2}\cos(\frac{\pi}{4})} = \frac{(\sqrt{2})^2}{\sqrt{2} \cdot \frac{1}{\sqrt{2}}} = \frac{2}{1} = 2$$

Answer: 2 (Option a).

7. **Question:** Evaluate the limit:

$$\lim_{x \to \infty} \frac{\sin \frac{2}{3^x}}{\sin \frac{3}{2^x}}$$

Solution: As $x \to \infty$, both $2/3^x \to 0$ and $3/2^x \to 0$. The limit is of the form $\frac{0}{0}$. We use the approximation $\sin\theta \approx \theta$ for $\theta \to 0$:

$$L = \lim_{x\to\infty} \frac{\frac{2}{3^x}}{\frac{3}{2^x}} = \lim_{x\to\infty} \frac{2}{3} \cdot \frac{2^x}{3^x} = \lim_{x\to\infty} \frac{2}{3} \cdot \left(\frac{2}{3}\right)^x$$

Since $2/3 < 1$, $\lim_{x\to\infty} \left(\frac{2}{3}\right)^x = 0$.

$$L = \frac{2}{3} \cdot 0 = 0$$

Correction Note: Since 0 is not an option, and $\frac{2}{3}$ is, the question likely intended $x \to 0$ with x multiplied in the argument, e.g., $\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$, which equals $2/3$. Based on the options, we assume a typographical error and that $x \to 0$ was intended. We solve the corrected problem:

$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)} = \lim_{x\to 0} \frac{\frac{\sin(2x)}{2x} \cdot 2x}{\frac{\sin(3x)}{3x} \cdot 3x} = \frac{1 \cdot 2x}{1 \cdot 3x} = \frac{2}{3}$$

Answer: $\frac{2}{3}$ (Option b).

8. **Question:** Evaluate the limit:

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

Solution: This limit requires the Maclaurin (Taylor) series expansion for $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The numerator is:

$$N = \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) - x + \frac{x^3}{6}$$

$$N = \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

The limit is:

$$L = \lim_{x\to 0} \frac{\frac{x^5}{120} - \frac{x^7}{5040} + \cdots}{x^5} = \lim_{x\to 0} \left(\frac{1}{120} - \frac{x^2}{5040} + \cdots\right)$$

$$L = \frac{1}{120}$$

Answer: $\frac{1}{120}$ (Option b).

9. **Question:** Evaluate the limit:

$$\lim_{x\to 0} \frac{(1+x)^5 - 1}{3x + 5x^2}$$

Solution: The limit is of the form $\frac{0}{0}$. We can use L'Hôpital's Rule or algebraic manipulation.

Method 1: L'Hôpital's Rule

$$L = \lim_{x\to 0} \frac{\frac{d}{dx}((1+x)^5 - 1)}{\frac{d}{dx}(3x + 5x^2)} = \lim_{x\to 0} \frac{5(1+x)^4}{3 + 10x}$$

Substituting $x = 0$:

$$L = \frac{5(1+0)^4}{3 + 10(0)} = \frac{5}{3}$$

Answer: $\frac{5}{3}$ (Option a).

10. **Question:** Evaluate the limit:

$$\lim_{\theta\to\frac{\pi}{4}} \frac{\sin\theta - \cos\theta}{\theta - \frac{\pi}{4}}$$

Solution: The limit is of the form $\frac{0}{0}$. We use L'Hôpital's Rule:

$$L = \lim_{\theta \to \frac{\pi}{4}} \frac{\frac{d}{d\theta}(\sin\theta - \cos\theta)}{\frac{d}{d\theta}(\theta - \frac{\pi}{4})} = \lim_{\theta \to \frac{\pi}{4}} \frac{\cos\theta + \sin\theta}{1}$$

Substituting $\theta = \frac{\pi}{4}$:

$$L = \cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Answer: $\sqrt{2}$ (Option a).

11. **Question:** Evaluate the limit (Let $a \neq 0$):

$$\lim_{x \to a} \left(\frac{\sin x}{\sin a} \right)^{\frac{1}{(x-a)}}$$

Solution: The limit is of the indeterminate form 1^∞. We use the formula $\lim f(x)^{g(x)} = e^{\lim g(x)(f(x)-1)}$.

$$L = e^{\lim_{x \to a} \frac{1}{x-a}\left(\frac{\sin x}{\sin a} - 1\right)}$$

$$L = e^{\lim_{x \to a} \frac{\sin x - \sin a}{(x-a)\sin a}}$$

We recognize the limit $\lim_{x \to a} \frac{\sin x - \sin a}{x - a}$ as the definition of the derivative of $\sin x$ at $x = a$, which is $\cos a$.

$$L = e^{\frac{\cos a}{\sin a}} = e^{\cot a}$$

Answer: $e^{\cot a}$ (Option a).

12. **Question:** Evaluate the limit:

$$\lim_{x \to 0} \frac{x}{\tan^{-1} 2x}$$

Solution: The limit is of the form $\frac{0}{0}$. We use the standard trigonometric limit approximation $\tan^{-1}\theta \approx \theta$ for $\theta \to 0$:

$$L = \lim_{x \to 0} \frac{x}{2x}$$

$$L = \frac{1}{2}$$

Answer: $\frac{1}{2}$ (Option b).

13. **Question:** Evaluate the limit:

$$\lim_{n \to \infty} (4^n + 5^n)^{\frac{1}{n}}$$

Solution: We factor out the largest base term, 5^n:

$$L = \lim_{n \to \infty} \left(5^n \left(\frac{4^n}{5^n} + 1 \right) \right)^{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} 5^{\frac{n}{n}} \cdot \left(\left(\frac{4}{5} \right)^n + 1 \right)^{\frac{1}{n}}$$

Since $4/5 < 1$, $\lim_{n \to \infty} \left(\frac{4}{5} \right)^n = 0$.

$$L = 5 \cdot (0 + 1)^0$$

$$L = 5 \cdot 1 = 5$$

Answer: 5 (Option b).

14. **Question:** Evaluate the limit:

$$\lim_{x \to \infty} \sqrt{\frac{x + \sin x}{x - \cos x}}$$

Solution: We divide the numerator and the denominator inside the square root by x:

$$L = \lim_{x\to\infty} \sqrt{\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}}$$

We use the Squeeze Theorem. Since $-1 \le \sin x \le 1$ and $-1 \le \cos x \le 1$, we have:

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\cos x}{x} = 0$$

Substituting these limits:

$$L = \sqrt{\frac{1+0}{1-0}} = 1$$

Answer: 1 (Option a).

15. **Question:** Evaluate the limit:

$$\lim_{n\to\infty} \frac{n^p \sin^2(n!)}{n^{1-p}(1+\frac{1}{n})}$$

for $p < 0$.

Solution: First, simplify the powers of n:

$$\frac{n^p}{n^{1-p}} = n^{p-(1-p)} = n^{2p-1}$$

Since $p < 0$, the exponent $2p-1$ is negative and $|2p-1| > 1$. Let $k = 1-2p > 1$. The term $n^{2p-1} = \dfrac{1}{n^k}$.

$$L = \lim_{n\to\infty} \frac{1}{n^k} \cdot \frac{\sin^2(n!)}{1+\frac{1}{n}}$$

We analyze the two factors:

- The first factor $\dfrac{1}{n^k} \to 0$ as $n \to \infty$.
- The second factor $\dfrac{\sin^2(n!)}{1+\frac{1}{n}}$ is bounded. Since $0 \le \sin^2(n!) \le 1$ and $1 < 1 + \dfrac{1}{n} \le 2$ (for $n \ge 1$), the fraction $\dfrac{\sin^2(n!)}{1+\frac{1}{n}}$ is bounded between 0 and 1.

The product of a term approaching zero and a bounded term is zero.

$$L = 0$$

Answer: 0 (Option b).

16. **Question:** Evaluate the limit:

$$\lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$$

Solution: The limit is of the form $\dfrac{0}{0}$. We multiply by the conjugate of the numerator:

$$L = \lim_{x\to 0} \frac{2-(1+\cos x)}{\sin^2 x(\sqrt{2}+\sqrt{1+\cos x})} = \lim_{x\to 0} \frac{1-\cos x}{\sin^2 x(\sqrt{2}+\sqrt{1+\cos x})}$$

We use the identities $1-\cos x = 2\sin^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$, so $\sin^2 x = 4\sin^2(x/2)\cos^2(x/2)$.

$$L = \lim_{x\to 0} \frac{2\sin^2(x/2)}{4\sin^2(x/2)\cos^2(x/2)(\sqrt{2}+\sqrt{1+\cos x})}$$

Cancel $2\sin^2(x/2)$:

$$L = \lim_{x\to 0} \frac{1}{2\cos^2(x/2)(\sqrt{2}+\sqrt{1+\cos x})}$$

Substitute $x = 0$:

$$L = \frac{1}{2\cos^2(0)(\sqrt{2}+\sqrt{1+\cos 0})} = \frac{1}{2(1)^2(\sqrt{2}+\sqrt{2})} = \frac{1}{2(2\sqrt{2})} = \frac{1}{4\sqrt{2}}$$

Answer: $\dfrac{1}{4\sqrt{2}}$ (Option b).

Integer Type Questions Solutions

17. **Question:** Evaluate the limit:

$$\lim_{n\to\infty} n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$$

Solution: We use the double angle identity $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$. Let $\theta = \frac{\pi}{4n}$.

$$L = \lim_{n\to\infty} n\cdot\frac{1}{2}\sin\left(2\cdot\frac{\pi}{4n}\right) = \lim_{n\to\infty}\frac{n}{2}\sin\left(\frac{\pi}{2n}\right)$$

We rewrite the expression to use the standard limit $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$. Let $\theta' = \frac{\pi}{2n}$. As $n\to\infty$, $\theta'\to 0$.

$$L = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{\sin(\frac{\pi}{2n})}{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{\sin(\frac{\pi}{2n})}{\frac{\pi}{2n}}\cdot\frac{\pi}{2}$$

$$L = \frac{1}{2}\cdot 1\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

Answer: $\pi/4$

18. **Question:** Evaluate the limit:

$$\lim_{x\to 3}\frac{(x^3+27)\ln(x-2)}{x^2-9}$$

Solution: The limit is of the indeterminate form $\frac{0}{0}$. We factor the polynomial terms:

$$x^3+27 = (x+3)(x^2-3x+9)$$

$$x^2-9 = (x-3)(x+3)$$

$$L = \lim_{x\to 3}\frac{(x+3)(x^2-3x+9)\ln(x-2)}{(x-3)(x+3)}$$

We cancel $(x+3)$ since $x\to 3$ implies $x\neq -3$.

$$L = \lim_{x\to 3}\frac{(x^2-3x+9)\ln(x-2)}{x-3}$$

We split the limit into two parts:

$$L = \lim_{x\to 3}(x^2-3x+9)\cdot\lim_{x\to 3}\frac{\ln(x-2)}{x-3}$$

- First limit: $3^2-3(3)+9 = 9-9+9 = 9$.
- Second limit: Let $y = x-3$. As $x\to 3$, $y\to 0$. The expression becomes $\frac{\ln((y+3)-2)}{y} = \frac{\ln(1+y)}{y}$.

$$\lim_{y\to 0}\frac{\ln(1+y)}{y} = 1$$

$$L = 9\cdot 1 = 9$$

Answer: 9

19. **Question:** Evaluate the limit:

$$\lim_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}$$

Find the value of the limit as a fraction in simplest form p/q, and give the value of $p+q$.

Solution: The limit is of the form $\frac{0}{0}$. We use the substitution $x = \frac{\pi}{2} - h$. As $x \to \frac{\pi}{2}$, $h \to 0$. The denominator: $(\pi - 2x)^3 = (\pi - 2(\frac{\pi}{2} - h))^3 = (\pi - \pi + 2h)^3 = (2h)^3 = 8h^3$. The numerator:

$$N = \cot(\frac{\pi}{2} - h) - \cos(\frac{\pi}{2} - h) = \tan h - \sin h$$

Factor the numerator:

$$N = \sin h \left(\frac{1}{\cos h} - 1 \right) = \sin h \left(\frac{1 - \cos h}{\cos h} \right)$$

We use the standard approximations $\sin h \approx h$, $1 - \cos h \approx \frac{h^2}{2}$, and $\cos h \approx 1$.

$$L = \lim_{h \to 0} \frac{\sin h \cdot (1 - \cos h)}{8h^3 \cos h} = \lim_{h \to 0} \frac{h \cdot \frac{h^2}{2}}{8h^3 \cdot 1} = \lim_{h \to 0} \frac{h^3/2}{8h^3}$$

$$L = \frac{1/2}{8} = \frac{1}{16}$$

The limit is $\frac{1}{16}$, so $p = 1$ and $q = 16$.

$$p + q = 1 + 16 = 17$$

Answer: 17

20. **Question:** Evaluate the limit:

$$\lim_{x \to 0} \frac{\sin[\cos x]}{1 + [\cos x]}, \text{ where [.] denotes the greatest integer function.}$$

Solution: We analyze the greatest integer function $[\cos x]$ as $x \to 0$:

- As $x \to 0$, $\cos x \to 1$.
- Since $\cos x$ is an even function and has a maximum at $x = 0$, for $x \neq 0$ near 0, we have $0 < \cos x < 1$.
- Therefore, $[\cos x] = 0$ for $x \neq 0$ near 0.

Substituting $[\cos x] = 0$:

$$L = \lim_{x \to 0} \frac{\sin(0)}{1 + 0} = \frac{0}{1} = 0$$

Answer: 0

21. **Question:** Evaluate the limit:

$$\lim_{x \to 0} \left[\frac{\sin x}{x} \right] \text{ where [.] denotes the greatest integer function.}$$

Solution: We know the fundamental limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

We need to know how the value approaches 1:

- Using the series expansion $\sin x = x - \frac{x^3}{6} + \cdots$, for $x \neq 0$ near 0, we have $\sin x < x$.
- Therefore, $\frac{\sin x}{x} < 1$.

Since $\frac{\sin x}{x}$ approaches 1 from the left (e.g., 0.9999), the greatest integer value is:

$$L = \left[\lim_{x \to 0} \frac{\sin x}{x} \right]^- = [1^-] = 0$$

Answer: 0