

★ Set 2 - Limits and Continuity

✓ Subjective Questions

1. Prove that $\lim_{x \rightarrow \infty} \frac{(3x-1)(2x+5)}{(x-3)(3x+7)} = 2$

Solution:

$$L = \lim_{x \rightarrow \infty} \frac{(3x-1)(2x+5)}{(x-3)(3x+7)}$$

Factor out x from each term in the numerator and the denominator:

$$L = \lim_{x \rightarrow \infty} \frac{x(3 - \frac{1}{x}) \cdot x(2 + \frac{5}{x})}{x(1 - \frac{3}{x}) \cdot x(3 + \frac{7}{x})} = \lim_{x \rightarrow \infty} \frac{x^2(3 - \frac{1}{x})(2 + \frac{5}{x})}{x^2(1 - \frac{3}{x})(3 + \frac{7}{x})}$$

Cancel x^2 and apply the limit, noting that $\lim_{x \rightarrow \infty} \frac{c}{x} = 0$ for any constant c :

$$L = \lim_{x \rightarrow \infty} \frac{(3 - \frac{1}{x})(2 + \frac{5}{x})}{(1 - \frac{3}{x})(3 + \frac{7}{x})} = \frac{(3-0)(2+0)}{(1-0)(3+0)} = \frac{3 \cdot 2}{1 \cdot 3} = \frac{6}{3} = 2$$

Thus, $\lim_{x \rightarrow \infty} \frac{(3x-1)(2x+5)}{(x-3)(3x+7)} = 2$. □

2. Show that $f(x) = \frac{1}{1+e^{\frac{1}{x}}}$ when $x \neq 0$, $= 0$ when $x = 0$ is not continuous at $x = 0$

Solution: For $f(x)$ to be continuous at $x = 0$, we must have $\lim_{x \rightarrow 0} f(x) = f(0)$. Given $f(0) = 0$. We need to check the left-hand limit (LHL) and the right-hand limit (RHL) at $x = 0$.

Right-Hand Limit (RHL): Let $x \rightarrow 0^+$.

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{1 + e^{\frac{1}{x}}}$$

As $x \rightarrow 0^+$, $\frac{1}{x} \rightarrow \infty$, so $e^{\frac{1}{x}} \rightarrow \infty$.

$$\lim_{x \rightarrow 0^+} f(x) = \frac{1}{1 + \infty} = \frac{1}{\infty} = 0$$

Left-Hand Limit (LHL): Let $x \rightarrow 0^-$.

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{1 + e^{\frac{1}{x}}}$$

As $x \rightarrow 0^-$, $\frac{1}{x} \rightarrow -\infty$, so $e^{\frac{1}{x}} \rightarrow 0$.

$$\lim_{x \rightarrow 0^-} f(x) = \frac{1}{1 + 0} = 1$$

Since LHL = 1 and RHL = 0, the limit $\lim_{x \rightarrow 0} f(x)$ does not exist. Alternatively, even though RHL = $f(0)$, the function is not continuous because LHL $\neq f(0)$. Since $\lim_{x \rightarrow 0} f(x)$ does not exist, $f(x)$ is ****not continuous**** at $x = 0$. □

3. Discuss the continuity of the function $f(x)$ defined by $f(x) = \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, x \neq 0$,
 $f(0) = -1$

Solution: The function is continuous for all $x \neq 0$ since it is a composition of elementary functions (exponential and rational functions), and the denominator $e^{\frac{1}{x}} + 1$ is never zero. We check continuity at $x = 0$. For continuity, we need $\lim_{x \rightarrow 0} f(x) = f(0) = -1$.

Right-Hand Limit (RHL): Let $x \rightarrow 0^+$.

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

As $x \rightarrow 0^+$, $\frac{1}{x} \rightarrow \infty$, so $e^{\frac{1}{x}} \rightarrow \infty$. We divide the numerator and denominator by $e^{\frac{1}{x}}$:

$$\lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \lim_{x \rightarrow 0^+} \frac{1 - e^{-\frac{1}{x}}}{1 + e^{-\frac{1}{x}}} = \frac{1 - 0}{1 + 0} = 1 \quad (\text{since } e^{-\frac{1}{x}} \rightarrow 0 \text{ as } x \rightarrow 0^+)$$

Left-Hand Limit (LHL): Let $x \rightarrow 0^-$.

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

As $x \rightarrow 0^-$, $\frac{1}{x} \rightarrow -\infty$, so $e^{\frac{1}{x}} \rightarrow 0$.

$$\lim_{x \rightarrow 0^-} f(x) = \frac{0 - 1}{0 + 1} = -1$$

Since LHL = -1 and RHL = 1 , the limit $\lim_{x \rightarrow 0} f(x)$ does not exist, because LHL \neq RHL. However, the LHL equals $f(0)$: LHL = $-1 = f(0)$. The RHL does not equal $f(0)$: RHL = $1 \neq -1$.

Since $\lim_{x \rightarrow 0} f(x)$ does not exist, $f(x)$ is ****not continuous**** at $x = 0$. It has a ****jump discontinuity**** at $x = 0$. Thus, $f(x)$ is continuous everywhere except at $x = 0$. \square

4. Show that $\lim_{x \rightarrow -2} \frac{\tan \pi x}{x+2} + \lim_{x \rightarrow \infty} (1 + \frac{1}{x^2})^x > 3$

Solution: Let $L_1 = \lim_{x \rightarrow -2} \frac{\tan \pi x}{x+2}$ and $L_2 = \lim_{x \rightarrow \infty} (1 + \frac{1}{x^2})^x$. We evaluate each limit separately.

Evaluating L_1 : Let $y = x + 2$. As $x \rightarrow -2$, $y \rightarrow 0$. So $x = y - 2$.

$$L_1 = \lim_{y \rightarrow 0} \frac{\tan(\pi(y - 2))}{y} = \lim_{y \rightarrow 0} \frac{\tan(\pi y - 2\pi)}{y}$$

Since $\tan(\theta - 2\pi) = \tan \theta$, we have:

$$L_1 = \lim_{y \rightarrow 0} \frac{\tan(\pi y)}{y}$$

Using the standard limit $\lim_{\theta \rightarrow 0} \frac{\tan k\theta}{\theta} = k$:

$$L_1 = \lim_{y \rightarrow 0} \frac{\tan(\pi y)}{\pi y} \cdot \pi = 1 \cdot \pi = \pi$$

Evaluating L_2 : This is of the indeterminate form 1^∞ . We use the formula $\lim_{x \rightarrow a} [f(x)]^{g(x)} = e^{\lim_{x \rightarrow a} g(x)[f(x)-1]}$.

$$L_2 = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x^2}\right)^x = e^{\lim_{x \rightarrow \infty} x \left(1 + \frac{1}{x^2} - 1\right)}$$

$$L_2 = e^{\lim_{x \rightarrow \infty} x \cdot \frac{1}{x^2}} = e^{\lim_{x \rightarrow \infty} \frac{1}{x}} = e^0 = 1$$

Checking the inequality: The sum is $L_1 + L_2 = \pi + 1$. We know that $\pi \approx 3.14159$.

$$L_1 + L_2 = \pi + 1 \approx 3.14159 + 1 = 4.14159$$

Since $4.14159 > 3$, the inequality $\lim_{x \rightarrow -2} \frac{\tan \pi x}{x+2} + \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x^2}\right)^x > 3$ is true. \square

5. Evaluate $\lim_{x \rightarrow \frac{\pi}{3}} \frac{\tan^3 x - 3 \tan x}{\cos(x + \frac{\pi}{6})}$

Solution: Let L be the required limit.

$$L = \lim_{x \rightarrow \frac{\pi}{3}} \frac{\tan^3 x - 3 \tan x}{\cos(x + \frac{\pi}{6})}$$

Numerator: Substitute $x = \frac{\pi}{3}$: $\tan^3(\frac{\pi}{3}) - 3 \tan(\frac{\pi}{3}) = (\sqrt{3})^3 - 3\sqrt{3} = 3\sqrt{3} - 3\sqrt{3} = 0$. **Denominator:** Substitute $x = \frac{\pi}{3}$: $\cos(\frac{\pi}{3} + \frac{\pi}{6}) = \cos(\frac{2\pi+\pi}{6}) = \cos(\frac{3\pi}{6}) = \cos(\frac{\pi}{2}) = 0$.

The limit is of the indeterminate form $\frac{0}{0}$, so we can use L'Hôpital's Rule or algebraic manipulation.

Method 1: L'Hôpital's Rule Let $N(x) = \tan^3 x - 3 \tan x$ and $D(x) = \cos(x + \frac{\pi}{6})$.

$$N'(x) = 3 \tan^2 x \cdot \sec^2 x - 3 \sec^2 x = 3 \sec^2 x (\tan^2 x - 1)$$

$$D'(x) = -\sin(x + \frac{\pi}{6})$$

$$L = \lim_{x \rightarrow \frac{\pi}{3}} \frac{N'(x)}{D'(x)} = \frac{3 \sec^2(\frac{\pi}{3})(\tan^2(\frac{\pi}{3}) - 1)}{-\sin(\frac{\pi}{3} + \frac{\pi}{6})}$$

We use $\sec(\frac{\pi}{3}) = 2$, $\tan(\frac{\pi}{3}) = \sqrt{3}$, and $\sin(\frac{\pi}{3} + \frac{\pi}{6}) = \sin(\frac{\pi}{2}) = 1$.

$$L = \frac{3(2)^2((\sqrt{3})^2 - 1)}{-1} = \frac{3 \cdot 4(3 - 1)}{-1} = \frac{12 \cdot 2}{-1} = -24$$

Method 2: Algebraic Manipulation Factor the numerator: $N(x) = \tan x(\tan^2 x - 3)$. Substitute $x = \frac{\pi}{3} + h$. As $x \rightarrow \frac{\pi}{3}$, $h \rightarrow 0$.

$$D(x) = \cos\left(\frac{\pi}{3} + h + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2} + h\right) = -\sin h$$

The limit becomes:

$$L = \lim_{h \rightarrow 0} \frac{\tan(\frac{\pi}{3} + h)(\tan^2(\frac{\pi}{3} + h) - 3)}{-\sin h}$$

Since $\tan(\frac{\pi}{3} + h)$ and $\tan^2(\frac{\pi}{3} + h)$ are continuous at $h = 0$:

$$L = \frac{\tan(\frac{\pi}{3})(\tan^2(\frac{\pi}{3}) - 3)}{-\lim_{h \rightarrow 0} \frac{\sin h}{h}} \cdot \lim_{h \rightarrow 0} \frac{h}{-\sin h}$$

This is incorrect as the term $\tan^2(\frac{\pi}{3} + h) - 3$ still goes to 0 as $h \rightarrow 0$. Let's go back to L'Hopital or rewrite the numerator.

We use the identity $\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$. So $\tan^3 x - 3 \tan x = -(3 \tan x - \tan^3 x) = -(1 - 3 \tan^2 x) \tan 3x$.

$$L = \lim_{x \rightarrow \frac{\pi}{3}} \frac{-(3 \tan x - \tan^3 x)}{\cos(x + \frac{\pi}{6})} = \lim_{x \rightarrow \frac{\pi}{3}} \frac{(1 - 3 \tan^2 x) \tan 3x}{\cos(x + \frac{\pi}{6})}$$

Let $x = \frac{\pi}{3} + h$. As $x \rightarrow \frac{\pi}{3}$, $h \rightarrow 0$.

$$\cos(x + \frac{\pi}{6}) = \cos(\frac{\pi}{3} + h + \frac{\pi}{6}) = \cos(\frac{\pi}{2} + h) = -\sin h$$

$$\tan 3x = \tan 3(\frac{\pi}{3} + h) = \tan(\pi + 3h) = \tan 3h$$

$$1 - 3 \tan^2 x \rightarrow 1 - 3 \tan^2(\frac{\pi}{3}) = 1 - 3(\sqrt{3})^2 = 1 - 9 = -8 \quad \text{as } x \rightarrow \frac{\pi}{3}$$

$$L = \lim_{h \rightarrow 0} \frac{-(1 - 3 \tan^2(\frac{\pi}{3} + h)) \tan 3h}{-\sin h} = \frac{-(1 - 3 \tan^2(\frac{\pi}{3})) \lim_{h \rightarrow 0} \frac{\tan 3h}{h} \cdot \lim_{h \rightarrow 0} \frac{h}{\sin h}}{-1}$$

$$L = \frac{-(1 - 3(3)) \cdot 3 \cdot 1}{-1} = \frac{-(-8) \cdot 3}{-1} = \frac{24}{-1} = -24$$

The limit is -24 . □

★ Multiple Choice Questions

6. If $f(2) = 4$ and $f'(2) = 1$ then $\lim_{x \rightarrow 2} \frac{xf(2) - 2f(x)}{x-2}$ equals:

- (a) 0
- (b) 2
- (c) 4
- (d) 6

Solution: Substituting $x = 2$ in the limit expression gives $\frac{2f(2) - 2f(2)}{2-2} = \frac{0}{0}$, so we use L'Hôpital's Rule.

$$L = \lim_{x \rightarrow 2} \frac{\frac{d}{dx}[xf(2) - 2f(x)]}{\frac{d}{dx}[x - 2]}$$

Since $f(2)$ is a constant, its derivative is 0.

$$L = \lim_{x \rightarrow 2} \frac{f(2) \cdot 1 - 2f'(x)}{1} = f(2) - 2f'(2)$$

Given $f(2) = 4$ and $f'(2) = 1$:

$$L = 4 - 2(1) = 4 - 2 = 2$$

Alternatively

$$L = \lim_{x \rightarrow 2} \frac{xf(2) - 2f(x)}{x - 2} = \lim_{x \rightarrow 2} \frac{xf(2) - 2f(2) + 2f(2) - 2f(x)}{x - 2}$$

$$L = \lim_{x \rightarrow 2} \frac{f(2)(x - 2)}{x - 2} - \lim_{x \rightarrow 2} \frac{2(f(x) - f(2))}{x - 2}$$

$$L = f(2) - 2 \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$$

By the definition of the derivative, $\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = f'(2)$.

$$L = f(2) - 2f'(2) = 4 - 2(1) = 2$$

The correct option is (b). □

7. Find the value of k if $\lim_{x \rightarrow 0} \frac{\log(a+x) - \log a}{x} + k \lim_{x \rightarrow e} \frac{\log x - 1}{x - e} = 1$:

- (a) $e[1 - \frac{1}{a}]$
- (b) $e[1 + \frac{1}{a}]$
- (c) $e[\frac{1}{a} - 1]$
- (d) $e[2 - \frac{1}{a}]$

Solution: Let $L_1 = \lim_{x \rightarrow 0} \frac{\log(a+x) - \log a}{x}$ and $L_2 = \lim_{x \rightarrow e} \frac{\log x - 1}{x - e}$.

Evaluate L_1 : This is the definition of the derivative of $f(x) = \log x$ at $x = a$.

$$L_1 = f'(a) \quad \text{where } f(x) = \log x$$

Since $f'(x) = \frac{1}{x}$, we have $L_1 = \frac{1}{a}$. (Using L'Hôpital's Rule: $L_1 = \lim_{x \rightarrow 0} \frac{\frac{1}{a+x}}{\frac{1}{1}} = \frac{1}{a}$).

Evaluate L_2 : This is the definition of the derivative of $g(x) = \log x$ at $x = e$. Note that $\log e = 1$.

$$L_2 = \lim_{x \rightarrow e} \frac{\log x - \log e}{x - e} = g'(e)$$

Since $g'(x) = \frac{1}{x}$, we have $L_2 = \frac{1}{e}$. (Using L'Hôpital's Rule: $L_2 = \lim_{x \rightarrow e} \frac{\frac{1}{x}}{\frac{1}{1}} = \frac{1}{e}$).

The given equation is $L_1 + kL_2 = 1$:

$$\frac{1}{a} + k \left(\frac{1}{e} \right) = 1$$

$$\frac{k}{e} = 1 - \frac{1}{a} = \frac{a-1}{a}$$

$$k = e \left(\frac{a-1}{a} \right) = e \left(1 - \frac{1}{a} \right)$$

The correct option is (a). □

8. Let $f : R \rightarrow R$ be a differentiable function having $f(2) = 6$, $f'(2) = \frac{1}{48}$. The value of $\lim_{x \rightarrow 2} \frac{\int_6^{f(x)} 4t^3 dt}{x-2}$ is:

- (a) 12
- (b) 18
- (c) 24
- (d) 36

Solution: Let L be the required limit.

$$L = \lim_{x \rightarrow 2} \frac{\int_6^{f(x)} 4t^3 dt}{x-2}$$

Substitute $x = 2$: $\frac{\int_6^{f(2)} 4t^3 dt}{2-2} = \frac{\int_6^6 4t^3 dt}{0} = \frac{0}{0}$. It is an indeterminate form, so we use L'Hôpital's Rule.

We use the Fundamental Theorem of Calculus (Leibniz's Rule) for the numerator's derivative:

$$\frac{d}{dx} \int_a^{g(x)} F(t) dt = F(g(x)) \cdot g'(x)$$

Here $F(t) = 4t^3$, $g(x) = f(x)$, and $a = 6$.

$$\frac{d}{dx} \left[\int_6^{f(x)} 4t^3 dt \right] = 4(f(x))^3 \cdot f'(x)$$

$$L = \lim_{x \rightarrow 2} \frac{4(f(x))^3 f'(x)}{1} = 4(f(2))^3 f'(2)$$

Given $f(2) = 6$ and $f'(2) = \frac{1}{48}$:

$$L = 4(6)^3 \left(\frac{1}{48} \right) = 4(216) \left(\frac{1}{48} \right) = \frac{4 \cdot 216}{48} = \frac{216}{12} = 18$$

Wait, let me re-evaluate $4(6)^3 \cdot \frac{1}{48}$

$$4 \cdot 6^3 \cdot \frac{1}{48} = 4 \cdot 216 \cdot \frac{1}{48} = 4 \cdot 6 \cdot 36 \cdot \frac{1}{48} = 24 \cdot 36 \cdot \frac{1}{48} = \frac{36}{2} = 18$$

The correct option is (b). □

9. Given $f'(2) = 6$ and $f'(1) = 4$ then $\lim_{h \rightarrow 0} \frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$:

- (a) does not exist
- (b) is equal to $\frac{-3}{2}$
- (c) **equal to 3**
- (d) equal to 2

Solution: Let L be the required limit. Substituting $h = 0$ gives $\frac{f(2)-f(2)}{f(1)-f(1)} = \frac{0}{0}$. We use L'Hôpital's Rule.

$$L = \lim_{h \rightarrow 0} \frac{\frac{d}{dh}[f(2h+2+h^2) - f(2)]}{\frac{d}{dh}[f(h-h^2+1) - f(1)]}$$

Using the chain rule: $\frac{d}{dh}f(g(h)) = f'(g(h)) \cdot g'(h)$.

$$\frac{d}{dh}[f(2h+2+h^2)] = f'(2h+2+h^2) \cdot (2+2h)$$

$$\frac{d}{dh}[f(h-h^2+1)] = f'(h-h^2+1) \cdot (1-2h)$$

$$L = \lim_{h \rightarrow 0} \frac{f'(2h+2+h^2) \cdot (2+2h)}{f'(h-h^2+1) \cdot (1-2h)}$$

As $h \rightarrow 0$, $2h+2+h^2 \rightarrow 2$ and $h-h^2+1 \rightarrow 1$.

$$L = \frac{f'(2) \cdot (2+0)}{f'(1) \cdot (1-0)} = \frac{2f'(2)}{f'(1)}$$

Given $f'(2) = 6$ and $f'(1) = 4$:

$$L = \frac{2(6)}{4} = \frac{12}{4} = 3$$

The correct option is (c). □

10. If $f(x) = \cot^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ and $g(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ then $\lim_{x \rightarrow 0} \frac{f(x)-f(a)}{g(x)-g(a)}$, $0 < a < \frac{1}{2}$ is:

- (a) $\frac{-3}{2x}$
- (b) $\frac{-3(1+x)}{2}$
- (c) $\frac{-3}{2}$
- (d) $\frac{-3(1-a^2)}{2}$

Solution: Let L be the required limit. Substitute $x = \tan \theta$ in $f(x)$. Since $0 < a < \frac{1}{2}$, and the limit is $x \rightarrow 0$, we consider x near 0, where $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$. In this range, let $x = \tan \theta$, where $\theta \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$.

$$f(x) = \cot^{-1}\left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}\right) = \cot^{-1}(\tan 3\theta) = \cot^{-1}\left(\cot\left(\frac{\pi}{2} - 3\theta\right)\right)$$

Since $\theta \in (-\frac{\pi}{6}, \frac{\pi}{6})$, $3\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so $\frac{\pi}{2} - 3\theta \in (0, \pi)$. Thus, $f(x) = \frac{\pi}{2} - 3\theta = \frac{\pi}{2} - 3 \tan^{-1} x$.

$$f'(x) = \frac{d}{dx}(\frac{\pi}{2} - 3 \tan^{-1} x) = -3 \cdot \frac{1}{1+x^2}$$

Substitute $x = \tan \theta$ in $g(x)$.

$$g(x) = \cos^{-1}\left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}\right) = \cos^{-1}(\cos 2\theta)$$

Since $\theta \in (-\frac{\pi}{6}, \frac{\pi}{6})$, $2\theta \in (-\frac{\pi}{3}, \frac{\pi}{3})$. Thus, $g(x) = 2\theta = 2 \tan^{-1} x$.

$$g'(x) = \frac{d}{dx}(2 \tan^{-1} x) = 2 \cdot \frac{1}{1+x^2}$$

The limit can be written as:

$$L = \lim_{x \rightarrow 0} \frac{f(x) - f(a)}{g(x) - g(a)}$$

This is of the form $\frac{f(0)-f(a)}{g(0)-g(a)}$, which is **not** an indeterminate form $\frac{0}{0}$ if $0 \neq a$. The limit is of the form $\frac{0}{0}$ only if $x \rightarrow a$. The question has a typo, it must be $\lim_{x \rightarrow a}$. Assuming the limit is $\lim_{x \rightarrow a}$:

$$L' = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

This is $\frac{0}{0}$ and can be written as:

$$L' = \lim_{x \rightarrow a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}$$

$$L' = \frac{-3 \cdot \frac{1}{1+a^2}}{2 \cdot \frac{1}{1+a^2}} = -\frac{3}{2}$$

Assuming the original question $\lim_{x \rightarrow 0}$ is correct:

$$L = \frac{f(0) - f(a)}{g(0) - g(a)}$$

$$f(0) = \cot^{-1}(0) = \frac{\pi}{2}. \quad g(0) = \cos^{-1}(1) = 0.$$

$$L = \frac{\frac{\pi}{2} - (\frac{\pi}{2} - 3 \tan^{-1} a)}{0 - 2 \tan^{-1} a} = \frac{3 \tan^{-1} a}{-2 \tan^{-1} a} = -\frac{3}{2}$$

Both interpretations yield the same result. The correct option is (c). □

11. f is defined in $[-5, 5]$ as $f(x) = x$ if x is rational, $= -x$ if x is irrational. Then:

- (a) $f(x)$ is continuous at every x , except $x = 0$
- (b) $f(x)$ is discontinuous at every x , except $x = 0$
- (c) $f(x)$ is continuous everywhere
- (d) $f(x)$ is discontinuous everywhere

Solution: A function $f(x)$ is continuous at $x = c$ if $\lim_{x \rightarrow c} f(x) = f(c)$.

Case 1: Continuity at $x = 0$ We need to show $\lim_{x \rightarrow 0} f(x) = f(0)$. If x is rational, $f(x) = x$. As $x \rightarrow 0$, $f(x) \rightarrow 0$. If x is irrational, $f(x) = -x$. As $x \rightarrow 0$, $f(x) \rightarrow 0$. So, $\lim_{x \rightarrow 0} f(x) = 0$. Since 0 is rational, $f(0) = 0$. Since $\lim_{x \rightarrow 0} f(x) = f(0)$, the function $f(x)$ is continuous at $x = 0$.

Case 2: Continuity at $x = c \neq 0$ For any $c \neq 0$, we consider the limit $\lim_{x \rightarrow c} f(x)$. There is a sequence of rational numbers $r_n \rightarrow c$ and a sequence of irrational numbers $i_n \rightarrow c$.

$$\lim_{n \rightarrow \infty} f(r_n) = \lim_{n \rightarrow \infty} r_n = c$$

$$\lim_{n \rightarrow \infty} f(i_n) = \lim_{n \rightarrow \infty} -i_n = -c$$

For the limit to exist, we must have $c = -c$, which implies $2c = 0$, so $c = 0$. Since we assumed $c \neq 0$, $c \neq -c$. Thus, $\lim_{x \rightarrow c} f(x)$ does not exist for $c \neq 0$. Therefore, $f(x)$ is discontinuous at every x except $x = 0$. The correct option is (b). \square

12. $f(x)$ and $g(x)$ are two differentiable functions on $[0, 2]$ such that $f''(x) - g''(x) = 0$, $f'(1) = 2g'(1) = 4$, $f(2) = 3g(2) = 9$ then $f(x) - g(x)$ at $x = \frac{3}{2}$:

- (a) 0
- (b) 4
- (c) 5
- (d) 3

Solution: Given $f''(x) - g''(x) = 0$, which implies $\frac{d^2}{dx^2}(f(x) - g(x)) = 0$. Let $h(x) = f(x) - g(x)$. Then $h''(x) = 0$. Integrating $h''(x) = 0$ twice gives:

$$h'(x) = A \quad (\text{where } A \text{ is a constant})$$

$$h(x) = Ax + B \quad (\text{where } B \text{ is a constant})$$

From $h'(x) = A$, we use the given conditions for the first derivatives. Given $f'(1) = 4$ and $2g'(1) = 4 \implies g'(1) = 2$.

$$h'(1) = f'(1) - g'(1) = 4 - 2 = 2$$

So, $A = 2$. Thus, $h(x) = 2x + B$.

Now we find B using the values at $x = 2$. Given $f(2) = 9$ and $3g(2) = 9 \implies g(2) = 3$.

$$h(2) = f(2) - g(2) = 9 - 3 = 6$$

Substitute into $h(x) = 2x + B$:

$$h(2) = 2(2) + B = 4 + B$$

Since $h(2) = 6$, we have $4 + B = 6$, so $B = 2$.

The function is $h(x) = f(x) - g(x) = 2x + 2$.

We need to find $f(x) - g(x)$ at $x = \frac{3}{2}$, which is $h(\frac{3}{2})$.

$$h\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right) + 2 = 3 + 2 = 5$$

The correct option is (c). □

13. Let $f(a) = g(a) = k$ and their n th derivatives $f^n(a)$, $g^n(a)$ exist and are not equal for some n . Further if $\lim_{x \rightarrow a} \frac{f(a)g(x) - f(a) - g(a)f(x) + f(a)}{g(x) - f(x)} = 4$ then the value of k is:

- (a) 4
- (b) 1
- (c) 2
- (d) -1

Solution: Let L be the required limit. Simplify the numerator:

$$N(x) = f(a)g(x) - f(a) - g(a)f(x) + f(a) = f(a)g(x) - g(a)f(x)$$

Since $f(a) = g(a) = k$, we have:

$$N(x) = kg(x) - kf(x) = k(g(x) - f(x))$$

The limit expression becomes:

$$L = \lim_{x \rightarrow a} \frac{k(g(x) - f(x))}{g(x) - f(x)}$$

Since $f^n(a) \neq g^n(a)$ for some n , it implies that $f(x)$ and $g(x)$ are not identical. In a small neighborhood around a , $g(x) - f(x) \neq 0$ (otherwise the functions would be identical). Thus, we can cancel the term $g(x) - f(x)$:

$$L = \lim_{x \rightarrow a} k = k$$

Given that $L = 4$, we have $k = 4$.

Wait, let's recheck the numerator $f(a)g(x) - f(a) - g(a)f(x) + f(a)$

$$N(x) = f(a)g(x) - g(a)f(x)$$

The limit $\lim_{x \rightarrow a} \frac{f(a)g(x) - g(a)f(x)}{g(x) - f(x)} = 4$. Substituting $f(a) = g(a) = k$:

$$L = \lim_{x \rightarrow a} \frac{kg(x) - kf(x)}{g(x) - f(x)} = \lim_{x \rightarrow a} \frac{k(g(x) - f(x))}{g(x) - f(x)} = k$$

So $k = 4$.

Let's check for a possible typo in the problem (a common one is $\frac{f(a)g(x)-f(a)f(x)-g(a)g(x)+g(a)f(x)}{g(x)-f(x)}$ which simplifies to $\frac{f(a)(g(x)-f(x))-g(a)(g(x)-f(x))}{g(x)-f(x)} = f(a) - g(a) = 0$ if $f(a) = g(a)$).

Consider the possibility that the simplification was intended to be:

$$N(x) = f(a)g(x) - f(a)f(x) - g(a)f(x) + f(a)f(x)$$

This is what the problem states: $N(x) = f(a)g(x) - f(a) - g(a)f(x) + f(a)$
It seems the terms $f(a)$ and $-f(a)$ cancel out:

$$N(x) = f(a)g(x) - g(a)f(x)$$

The only way the answer $k = 2$ is correct is if the expression was different, or if $k = 4$ is not an option. Since 4 is an option, we stick to $k = 4$.

****There must be a typo in the question or options. Assuming the intention was $k = 2$ and finding the expression.** If $k = 2$, the answer is (c). Since the provided solution set suggests $k = 2$, let's assume the correct simplification was intended to be $L = \frac{2k}{2} = k$. The only way to get a non-constant expression (in k) that simplifies to 4 would be if the terms didn't cancel so cleanly.**

Let's assume the question meant $f(a)g(x) - f(x)g(a)$ but one of the $f(a)$ was $f(x)$.

Let's assume the $N(x)$ was $f(a)g(x) - g(a)f(x)$ as calculated. So $k = 4$. There must be a typo in the options if $k = 2$ is the intended answer.

Final check on the provided expression: $f(a)g(x) - f(a) - g(a)f(x) + f(a)$. $N(x) = f(a)g(x) - g(a)f(x) = kg(x) - kf(x) = k(g(x) - f(x))$. $L = \frac{k(g(x)-f(x))}{g(x)-f(x)} = k$. Since $L = 4$, $k = 4$.

Conclusion: Based on the exact expression, $k = 4$. Assuming the provided answer key is correct and $k = 2$ is the answer, there is a typo in the question or options. Sticking to the mathematically correct result based on the text: $k = 4$. However, since 4 is an option, I will choose 4.

Wait, the options are: (a) 4, (b) 1, (c) 2, (d) -1. The calculation leads to $k = 4$. Since $k = 4$ is an option, let's stick with $k = 4$. The correct option is (a). \square

14. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{[1 - \tan(\frac{x}{2})][1 - \sin x]}{[1 + \tan(\frac{x}{2})][\pi - 2x]^3}$ is:

- (a) $\frac{1}{32}$
- (b) $\frac{1}{8}$
- (c) $\frac{1}{16}$
- (d) 0

Solution: Let $x = \frac{\pi}{2} + h$. As $x \rightarrow \frac{\pi}{2}$, $h \rightarrow 0$.

$$1 - \sin x = 1 - \sin\left(\frac{\pi}{2} + h\right) = 1 - \cos h$$

$$\pi - 2x = \pi - 2\left(\frac{\pi}{2} + h\right) = \pi - \pi - 2h = -2h$$

$$\tan\left(\frac{x}{2}\right) = \tan\left(\frac{\pi}{4} + \frac{h}{2}\right) = \frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{h}{2}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{h}{2}\right)} = \frac{1 + \tan\left(\frac{h}{2}\right)}{1 - \tan\left(\frac{h}{2}\right)}$$

Substitute into the limit L :

$$L = \lim_{h \rightarrow 0} \frac{\left[1 - \frac{1 + \tan\left(\frac{h}{2}\right)}{1 - \tan\left(\frac{h}{2}\right)}\right][1 - \cos h]}{\left[1 + \frac{1 + \tan\left(\frac{h}{2}\right)}{1 - \tan\left(\frac{h}{2}\right)}\right](-2h)^3}$$

Simplify the terms involving $\tan\left(\frac{h}{2}\right)$:

$$1 - \frac{1 + \tan\left(\frac{h}{2}\right)}{1 - \tan\left(\frac{h}{2}\right)} = \frac{1 - \tan\left(\frac{h}{2}\right) - (1 + \tan\left(\frac{h}{2}\right))}{1 - \tan\left(\frac{h}{2}\right)} = \frac{-2 \tan\left(\frac{h}{2}\right)}{1 - \tan\left(\frac{h}{2}\right)}$$

$$1 + \frac{1 + \tan\left(\frac{h}{2}\right)}{1 - \tan\left(\frac{h}{2}\right)} = \frac{1 - \tan\left(\frac{h}{2}\right) + 1 + \tan\left(\frac{h}{2}\right)}{1 - \tan\left(\frac{h}{2}\right)} = \frac{2}{1 - \tan\left(\frac{h}{2}\right)}$$

Substitute back:

$$L = \lim_{h \rightarrow 0} \frac{\left[\frac{-2 \tan\left(\frac{h}{2}\right)}{1 - \tan\left(\frac{h}{2}\right)}\right][1 - \cos h]}{\left[\frac{2}{1 - \tan\left(\frac{h}{2}\right)}\right](-8h^3)}$$

Cancel $\frac{1}{1 - \tan\left(\frac{h}{2}\right)}$ and simplify:

$$L = \lim_{h \rightarrow 0} \frac{-2 \tan\left(\frac{h}{2}\right)(1 - \cos h)}{2(-8h^3)} = \lim_{h \rightarrow 0} \frac{-2 \tan\left(\frac{h}{2}\right)(2 \sin^2 \frac{h}{2})}{-16h^3}$$

$$L = \lim_{h \rightarrow 0} \frac{4 \tan\left(\frac{h}{2}\right) \sin^2\left(\frac{h}{2}\right)}{16h^3} = \frac{1}{4} \lim_{h \rightarrow 0} \frac{\tan\left(\frac{h}{2}\right)}{h} \cdot \frac{\sin\left(\frac{h}{2}\right)}{h} \cdot \frac{\sin\left(\frac{h}{2}\right)}{h}$$

Using the standard limits $\lim_{h \rightarrow 0} \frac{\tan ah}{h} = a$ and $\lim_{h \rightarrow 0} \frac{\sin ah}{h} = a$:

$$L = \frac{1}{4} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$$

The correct option is (a). □

15. If

$$f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|} - \frac{1}{x}\right)}, & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

then $f(x)$ is:

(a) continuous for all x but not differentiable at $x = 0$

- (b) discontinuous everywhere
- (c) neither differentiable nor continuous at $x = 0$
- (d) **continuous everywhere**

Solution: For $x \neq 0$, $f(x)$ is a product of elementary functions, so it is continuous. We check continuity at $x = 0$. For continuity at $x = 0$, we need $\lim_{x \rightarrow 0} f(x) = f(0)$. Given $f(0) = 0$.

Case 1: Right-Hand Limit (RHL): $x \rightarrow 0^+$ For $x > 0$, $|x| = x$.

$$f(x) = xe^{-\left(\frac{1}{x} - \frac{1}{x}\right)} = xe^{-0} = x \cdot 1 = x$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$$

Case 2: Left-Hand Limit (LHL): $x \rightarrow 0^-$ For $x < 0$, $|x| = -x$.

$$f(x) = xe^{-\left(\frac{1}{-x} - \frac{1}{x}\right)} = xe^{-\left(-\frac{1}{x} - \frac{1}{x}\right)} = xe^{\frac{2}{x}}$$

Let $y = \frac{1}{x}$. As $x \rightarrow 0^-$, $y \rightarrow -\infty$. Also $x = \frac{1}{y}$.

$$\text{LHL} = \lim_{y \rightarrow -\infty} \frac{1}{y} e^{2y} = \lim_{y \rightarrow -\infty} \frac{e^{2y}}{y}$$

As $y \rightarrow -\infty$, $e^{2y} \rightarrow 0$ and $y \rightarrow -\infty$. This limit is 0 (exponential decay dominates linear increase). More formally, let $t = -2y$. As $y \rightarrow -\infty$, $t \rightarrow \infty$.

$$\text{LHL} = \lim_{t \rightarrow \infty} \frac{e^{-t}}{-t/2} = -2 \lim_{t \rightarrow \infty} \frac{1}{te^t} = -2 \cdot 0 = 0$$

Since $\text{LHL} = 0$, $\text{RHL} = 0$, and $f(0) = 0$, we have $\lim_{x \rightarrow 0} f(x) = f(0)$. Thus, $f(x)$ is ****continuous everywhere****.

****Check Differentiability at $x = 0$ (for completeness)****

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h)}{h}$$

Right-Hand Derivative (RHD): $h \rightarrow 0^+$

$$\text{RHD} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1$$

Left-Hand Derivative (LHD): $h \rightarrow 0^-$

$$\text{LHD} = \lim_{h \rightarrow 0^-} \frac{he^{\frac{2}{h}}}{h} = \lim_{h \rightarrow 0^-} e^{\frac{2}{h}}$$

As $h \rightarrow 0^-$, $\frac{2}{h} \rightarrow -\infty$, so $e^{\frac{2}{h}} \rightarrow 0$.

$$\text{LHD} = 0$$

Since $\text{LHD} \neq \text{RHD}$ ($0 \neq 1$), the function is ****not differentiable at $x = 0$ ****.

The function is continuous everywhere but not differentiable at $x = 0$. The correct option is (a). \square