Set 1 - Limits and Continuity

Subjective Questions - Solutions

1. If

$$f(x) = \begin{cases} \frac{e^{\frac{1}{(x-1)}}-2}{e^{\frac{1}{(x-1)}}+2}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$$

Discuss the behavior of $f(x)$ at $x = 1$.

Solution: To discuss the behavior at $x = 1$, we evaluate the Left Hand Limit (LHL) and Right Hand Limit (RHL) at $x = 1$ and compare them with $f(1)$.

Right Hand Limit (RHL) at $x = 1$: Let $x = 1 + h$, where $h \to 0^+$.

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0^+} \frac{e^{\frac{1}{((1+h)-1)}} - 2}{e^{\frac{1}{((1+h)-1)}} + 2} = \lim_{h\to 0^+} \frac{e^{\frac{1}{h}} - 2}{e^{\frac{1}{h}} + 2}$$

As $h \to 0^+$, $\frac{1}{h} \to \infty$, and $e^{\frac{1}{h}} \to \infty$. We divide the numerator and denominator by $e^{\frac{1}{h}}$:

$$\lim_{h\to 0^+} \frac{1 - 2e^{-\frac{1}{h}}}{1 + 2e^{-\frac{1}{h}}}$$

As $h \to 0^+$, $-\frac{1}{h} \to -\infty$, and $e^{-\frac{1}{h}} \to 0$.

$$\lim_{x\to 1^+} f(x) = \frac{1 - 2(0)}{1 + 2(0)} = 1$$

Left Hand Limit (LHL) at $x = 1$: Let $x = 1 - h$, where $h \to 0^+$.

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0^+} \frac{e^{\frac{1}{((1-h)-1)}} - 2}{e^{\frac{1}{((1-h)-1)}} + 2} = \lim_{h\to 0^+} \frac{e^{-\frac{1}{h}} - 2}{e^{-\frac{1}{h}} + 2}$$

As $h \to 0^+$, $-\frac{1}{h} \to -\infty$, and $e^{-\frac{1}{h}} \to 0$.

$$\lim_{x\to 1^-} f(x) = \frac{0 - 2}{0 + 2} = -1$$

Function Value:

$$f(1) = 1$$

Conclusion: Since $\lim_{x\to 1^+} f(x) = 1$ and $\lim_{x\to 1^-} f(x) = -1$, the LHL $\neq$ RHL. Thus, $\lim_{x\to 1} f(x)$ **does not exist**. The function $f(x)$ is **discontinuous** at $x = 1$ (specifically, a jump discontinuity).

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2. Evaluate $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^2 \sin x}$

Solution: As $x \to 0$, the expression is of the form $\frac{0}{0}$, so we can use L'Hôpital's Rule or Maclaurin Series expansion.

Method 1: Using L'Hôpital's Rule

$$L = \lim_{x \to 0} \frac{x \cos x - \sin x}{x^2 \sin x}$$

Applying L'Hôpital's Rule (differentiate numerator and denominator):

$$L = \lim_{x \to 0} \frac{(\cos x - x \sin x) - \cos x}{2x \sin x + x^2 \cos x} = \lim_{x \to 0} \frac{-x \sin x}{2x \sin x + x^2 \cos x}$$

For $x \neq 0$, we can divide the numerator and denominator by x:

$$L = \lim_{x \to 0} \frac{-\sin x}{2 \sin x + x \cos x}$$

The limit is still $\frac{0}{0}$. Applying L'Hôpital's Rule again:

$$L = \lim_{x \to 0} \frac{-\cos x}{2 \cos x + (\cos x - x \sin x)} = \lim_{x \to 0} \frac{-\cos x}{3 \cos x - x \sin x}$$

Now, substitute $x = 0$:

$$L = \frac{-\cos 0}{3 \cos 0 - 0 \sin 0} = \frac{-1}{3(1) - 0} = -\frac{1}{3}$$

Method 2: Using Standard Limits and Series Expansion We can use the standard limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and the Maclaurin series $\sin x = x - \frac{x^3}{3!} + O(x^5)$ and $\cos x = 1 - \frac{x^2}{2!} + O(x^4)$.

$$L = \lim_{x \to 0} \frac{x \cos x - \sin x}{x^2 \sin x} = \lim_{x \to 0} \frac{x \cos x - \sin x}{x^3 \cdot \frac{\sin x}{x}}$$

Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, we focus on the numerator:

$$L = \lim_{x \to 0} \frac{x \left(1 - \frac{x^2}{2!} + O(x^4)\right) - \left(x - \frac{x^3}{3!} + O(x^5)\right)}{x^3}$$

$$L = \lim_{x \to 0} \frac{\left(x - \frac{x^3}{2} + O(x^5)\right) - \left(x - \frac{x^3}{6} + O(x^5)\right)}{x^3}$$

$$L = \lim_{x \to 0} \frac{-\frac{x^3}{2} + \frac{x^3}{6} + O(x^5)}{x^3} = \lim_{x \to 0} \left(-\frac{1}{2} + \frac{1}{6} + O(x^2)\right)$$

$$L = -\frac{1}{2} + \frac{1}{6} = \frac{-3 + 1}{6} = -\frac{2}{6} = -\frac{1}{3}$$

The value of the limit is $-\mathbf{\frac{1}{3}}$.

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3. Let

$$f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2}, & \text{if } x < 0 \\ a, & \text{if } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}, & \text{if } x > 0 \end{cases}$$

Find the value of a so that the function may be continuous at $x = 0$.

Solution: For $f(x)$ to be continuous at $x = 0$, we must have:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

We are given $f(0) = a$.

Left Hand Limit (LHL) at $x = 0$:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{1 - \cos 4x}{x^2}$$

Using the identity $1 - \cos\theta = 2\sin^2(\frac{\theta}{2})$:

$$\lim_{x \to 0} \frac{2\sin^2(2x)}{x^2} = \lim_{x \to 0} 2 \cdot \frac{\sin(2x)}{x} \cdot \frac{\sin(2x)}{x}$$

To use the standard limit $\lim_{y \to 0} \frac{\sin y}{y} = 1$, we multiply and divide by 2:

$$\lim_{x \to 0} 2 \cdot \left(\frac{\sin(2x)}{2x} \cdot 2 \right) \cdot \left(\frac{\sin(2x)}{2x} \cdot 2 \right) = 2 \cdot (1 \cdot 2) \cdot (1 \cdot 2) = 8$$

Right Hand Limit (RHL) at $x = 0$:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}$$

This is of the form $\frac{0}{0}$. We rationalize the denominator:

$$\lim_{x \to 0} \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} \cdot \frac{\sqrt{16 + \sqrt{x}} + 4}{\sqrt{16 + \sqrt{x}} + 4}$$

$$= \lim_{x \to 0} \frac{\sqrt{x}(\sqrt{16 + \sqrt{x}} + 4)}{(16 + \sqrt{x}) - 4^2} = \lim_{x \to 0} \frac{\sqrt{x}(\sqrt{16 + \sqrt{x}} + 4)}{16 + \sqrt{x} - 16}$$

$$= \lim_{x \to 0} \frac{\sqrt{x}(\sqrt{16 + \sqrt{x}} + 4)}{\sqrt{x}}$$

For $x > 0$, we can cancel $\sqrt{x}$:

$$= \lim_{x \to 0} (\sqrt{16 + \sqrt{x}} + 4) = \sqrt{16 + \sqrt{0}} + 4 = \sqrt{16} + 4 = 4 + 4 = 8$$

For continuity at $x = 0$:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \implies 8 = 8 = a$$

The value of a is **8**.

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4. Evaluate the values of a, b and c such that $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$.

Solution: Let $L = \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x}$. The limit exists and is finite ($L = 2$).

As $x \to 0$, the denominator $x\sin x \to 0 \cdot 0 = 0$. For the limit to be finite, the numerator must also approach zero as $x \to 0$ (must be of the $\frac{0}{0}$ form).

Condition 1 (Numerator $\to 0$):

$$\lim_{x\to 0}(ae^x - b\cos x + ce^{-x}) = 0$$

$$ae^0 - b\cos 0 + ce^0 = 0$$

$$a(1) - b(1) + c(1) = 0 \implies \mathbf{a - b + c = 0} \quad (\mathbf{I})$$

Since the limit is of the $\frac{0}{0}$ form, we use L'Hôpital's Rule.

$$L = \lim_{x\to 0} \frac{\frac{d}{dx}(ae^x - b\cos x + ce^{-x})}{\frac{d}{dx}(x\sin x)} = \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x}$$

Now, as $x \to 0$, the denominator $\sin x + x\cos x \to 0 + 0 = 0$. For the limit to be 2, the new numerator must also approach zero.

Condition 2 (New Numerator $\to 0$):

$$\lim_{x\to 0}(ae^x + b\sin x - ce^{-x}) = 0$$

$$ae^0 + b\sin 0 - ce^0 = 0$$

$$a(1) + b(0) - c(1) = 0 \implies \mathbf{a - c = 0} \implies \mathbf{a = c} \quad (\mathbf{II})$$

Substitute (**II**) into (**I**):

$$a - b + a = 0 \implies 2a - b = 0 \implies \mathbf{b = 2a} \quad (\mathbf{III})$$

The limit is still of the $\frac{0}{0}$ form, so we apply L'Hôpital's Rule again:

$$L = \lim_{x\to 0} \frac{\frac{d}{dx}(ae^x + b\sin x - ce^{-x})}{\frac{d}{dx}(\sin x + x\cos x)} = \lim_{x\to 0} \frac{ae^x + b\cos x + ce^{-x}}{\cos x + (\cos x - x\sin x)}$$

$$L = \lim_{x\to 0} \frac{ae^x + b\cos x + ce^{-x}}{2\cos x - x\sin x}$$

Now, as $x \to 0$, the denominator $2\cos x - x\sin x \to 2(1) - 0 = 2 \neq 0$. We can substitute $x = 0$:

$$L = \frac{ae^0 + b\cos 0 + ce^0}{2\cos 0 - 0\sin 0} = \frac{a(1) + b(1) + c(1)}{2(1) - 0} = \frac{a + b + c}{2}$$

We are given that $L = 2$:

$$\frac{a + b + c}{2} = 2 \implies \mathbf{a + b + c = 4} \quad (\mathbf{IV})$$

We use (**II**) and (**III**) in (**IV**):

$$a + (2a) + a = 4$$

$$4a = 4 \implies \mathbf{a = 1}$$

From (**II**): $c = a \implies \mathbf{c = 1}$

From (**III**): $b = 2a \implies b = 2(1) \implies \mathbf{b = 2}$

The values are $\mathbf{a = 1}$, $\mathbf{b = 2}$, and $\mathbf{c = 1}$.

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5. Find the value of $f(0)$ so that the function $f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}, x \neq 0$ is continuous at $x = 0$.

Solution: For $f(x)$ to be continuous at $x = 0$, $f(0)$ must be equal to $\lim_{x\to 0} f(x)$.

$$f(0) = \lim_{x\to 0} \left(\frac{1}{x} - \frac{2}{e^{2x} - 1} \right)$$

We combine the fractions:

$$f(0) = \lim_{x\to 0} \frac{(e^{2x} - 1) - 2x}{x(e^{2x} - 1)}$$

As $x \to 0$, the numerator $\to (e^0 - 1) - 0 = 0$ and the denominator $\to 0(e^0 - 1) = 0$. The limit is of the $\frac{0}{0}$ form.

Method 1: Using L'Hôpital's Rule Apply L'Hôpital's Rule:

$$f(0) = \lim_{x\to 0} \frac{\frac{d}{dx}(e^{2x} - 1 - 2x)}{\frac{d}{dx}(xe^{2x} - x)} = \lim_{x\to 0} \frac{2e^{2x} - 2}{(1 \cdot e^{2x} + x \cdot 2e^{2x}) - 1} = \lim_{x\to 0} \frac{2e^{2x} - 2}{e^{2x} + 2xe^{2x} - 1}$$

The limit is still $\frac{0}{0}$. Apply L'Hôpital's Rule again:

$$f(0) = \lim_{x\to 0} \frac{\frac{d}{dx}(2e^{2x} - 2)}{\frac{d}{dx}(e^{2x} + 2xe^{2x} - 1)} = \lim_{x\to 0} \frac{4e^{2x}}{2e^{2x} + (2 \cdot e^{2x} + 2x \cdot 2e^{2x})}$$

$$f(0) = \lim_{x\to 0} \frac{4e^{2x}}{4e^{2x} + 4xe^{2x}}$$

Substitute $x = 0$:

$$f(0) = \frac{4e^0}{4e^0 + 4(0)e^0} = \frac{4(1)}{4(1) + 0} = \frac{4}{4} = 1$$

Method 2: Using Maclaurin Series Use the expansion $e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + O(x^4)$:

$$f(0) = \lim_{x\to 0} \frac{\left(1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + O(x^4)\right) - 1 - 2x}{x\left(\left(1 + 2x + \frac{4x^2}{2} + O(x^3)\right) - 1\right)}$$

$$f(0) = \lim_{x\to 0} \frac{2x^2 + \frac{4x^3}{3} + O(x^4)}{x\left(2x + 2x^2 + O(x^3)\right)}$$

$$f(0) = \lim_{x \to 0} \frac{2x^2 + \frac{4x^3}{3} + O(x^4)}{2x^2 + 2x^3 + O(x^4)}$$

Divide numerator and denominator by x^2:

$$f(0) = \lim_{x \to 0} \frac{2 + \frac{4x}{3} + O(x^2)}{2 + 2x + O(x^2)} = \frac{2 + 0 + 0}{2 + 0 + 0} = 1$$

The value of $f(0)$ is **1**.

Multiple Choice Questions - Solutions

6. The function $f(x) = |x| + \frac{|x|}{x}$ is:

   (a) discontinuous at the origin because $|x|$ is discontinuous there

   (b) **discontinuous at the origin because $\frac{|x|}{x}$ is discontinuous there**

   (c) discontinuous at -1 because $\frac{|x|}{x}$ is discontinuous there

   (d) continuous at the origin

   Solution: The function $f(x)$ is defined as:

$$f(x) = \begin{cases} x + \frac{x}{x} = x + 1, & \text{if } x > 0 \\ -x + \frac{-x}{x} = -x - 1, & \text{if } x < 0 \\ \text{undefined}, & \text{if } x = 0 \end{cases}$$

   Since $\frac{|x|}{x}$ is not defined at $x = 0$, $f(x)$ is not defined at $x = 0$. For the limit at $x = 0$:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1) = 1$$
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x - 1) = -1$$

   Since $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$, $\lim_{x \to 0} f(x)$ does not exist, and $f(x)$ is discontinuous at $x = 0$. The discontinuity arises because the component $\frac{|x|}{x}$ (which is 1 for $x > 0$ and -1 for $x < 0$) is discontinuous at $x = 0$.

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7. $\lim_{x \to \infty} \left(\frac{x+1}{x+2}\right)^{2x+1}$ is:

   (a) $\mathbf{e^{-2}}$

   (b) e

   (c) e^{-3}

   (d) e^2

   Solution: The limit is of the indeterminate form 1^∞ as $\lim_{x \to \infty} \frac{x+1}{x+2} = 1$ and $\lim_{x \to \infty}(2x+1) = \infty$. We use the standard formula: $\lim_{x \to a}[f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)(f(x)-1)}$.

$$L = e^{\lim_{x \to \infty} (2x+1)\left(\frac{x+1}{x+2} - 1\right)}$$

$$\frac{x+1}{x+2} - 1 = \frac{x + 1 - (x + 2)}{x + 2} = \frac{-1}{x + 2}$$

$$L = e^{\lim_{x\to\infty}(2x+1)\left(\frac{-1}{x+2}\right)} = e^{\lim_{x\to\infty}\frac{-2x-1}{x+2}}$$

To evaluate the limit in the exponent, we divide the numerator and denominator by x:

$$\lim_{x\to\infty}\frac{-2x-1}{x+2} = \lim_{x\to\infty}\frac{-2-\frac{1}{x}}{1+\frac{2}{x}} = \frac{-2-0}{1+0} = -2$$

Therefore, $L = \mathbf{e^{-2}}$.

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8. The value of $\lim_{n\to\infty}\left(\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\text{up to n terms}\right)$ is:

   (a) $\frac{-1}{2}$
   (b) 0
   (c) $\mathbf{\frac{1}{2}}$
   (d) $\frac{1}{3}$

   Solution: The k-th term of the series is $T_k = \frac{1}{(2k-1)(2k+1)}$. We use partial fraction decomposition:

$$T_k = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

   The sum of the first n terms, S_n, is a telescoping series:

$$S_n = \sum_{k=1}^{n} T_k = \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

$$S_n = \frac{1}{2}\left[1-\frac{1}{2n+1}\right]$$

   Now, we evaluate the limit as $n\to\infty$:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{2}\left[1-\frac{1}{2n+1}\right] = \frac{1}{2}[1-0] = \frac{1}{2}$$

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9. If $\lim_{x\to 0}(1+ax)^{\frac{b}{x}} = e^4$ where a and b are natural numbers then:

   (a) $\mathbf{ab = 4}$
   (b) $ab = 6$
   (c) $ab = 8$
   (d) $ab = 14$

   Solution: The limit is of the indeterminate form 1^∞. We use the standard formula: $\lim_{x\to 0}(1+f(x))^{g(x)} = e^{\lim_{x\to 0} f(x)g(x)}$, where $f(x) = ax$ and $g(x) = \frac{b}{x}$.

$$\lim_{x\to 0}(1+ax)^{\frac{b}{x}} = e^{\lim_{x\to 0}(ax)\left(\frac{b}{x}\right)}$$
$$= e^{\lim_{x\to 0}(ab)} = e^{ab}$$

We are given that $\lim_{x\to0}(1+ax)^{\frac{b}{x}} = e^4$.

$$e^{ab} = e^4$$

By comparing the exponents, we get $\mathbf{ab = 4}$.

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10. $\lim_{x\to0} \frac{a^x-1}{\sqrt{a+x}-\sqrt{a}}$ is:

 (a) $2\sqrt{a}\cdot 2\log a$
 (b) $2\sqrt{3a}\log a$
 (c) $\mathbf{2\sqrt{a}\log a}$
 (d) $\sqrt{a}\log a$

Solution: The limit is of the form $\frac{0}{0}$. We can use standard limits and rationalization.

$$L = \lim_{x\to0} \frac{a^x-1}{\sqrt{a+x}-\sqrt{a}}$$

We multiply the numerator and denominator by the conjugate of the denominator: $\sqrt{a+x}+\sqrt{a}$.

$$L = \lim_{x\to0} \frac{(a^x-1)(\sqrt{a+x}+\sqrt{a})}{(\sqrt{a+x}-\sqrt{a})(\sqrt{a+x}+\sqrt{a})}$$

$$L = \lim_{x\to0} \frac{(a^x-1)(\sqrt{a+x}+\sqrt{a})}{(a+x)-a} = \lim_{x\to0} \frac{(a^x-1)(\sqrt{a+x}+\sqrt{a})}{x}$$

We rearrange to use the standard limit $\lim_{x\to0} \frac{a^x-1}{x} = \log a$:

$$L = \lim_{x\to0}\left(\frac{a^x-1}{x}\right)\cdot \lim_{x\to0}(\sqrt{a+x}+\sqrt{a})$$

$$L = (\log a)\cdot(\sqrt{a+0}+\sqrt{a}) = (\log a)\cdot(2\sqrt{a})$$
$$L = 2\sqrt{a}\log a$$

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11. The function $f(x) = [x]\cos\left[\frac{2x-1}{2}\right]\pi$ where [.] denotes the greatest integer function is discontinuous at:

 (a) all x
 (b) no x
 (c) **all integral points**
 (d) x which is not an integer

Solution: We simplify the argument of the cosine function:

$$\left[\frac{2x-1}{2}\right] = \left[\frac{2x}{2}-\frac{1}{2}\right] = \left[x-\frac{1}{2}\right]$$

The function is $f(x) = [x]\cos\left(\left[x-\frac{1}{2}\right]\pi\right)$.

The greatest integer function $[y]$ is discontinuous at every integer n. Thus, $[x]$ is discontinuous at all integers, and $\left[x - \frac{1}{2}\right]$ is discontinuous when $x - \frac{1}{2}$ is an integer, i.e., $x = n + \frac{1}{2}$, where $n \in \mathbb{Z}$.

Let's check at an integer n:

- $\mathbf{x \to n^+}$: $x = n + h$, $h \to 0^+$.

$$\lim_{x \to n^+} f(x) = \lim_{h \to 0^+} [n + h] \cos\left(\left[n + h - \frac{1}{2}\right]\pi\right)$$

Since $h > 0$, $[n+h] = n$. Since $h - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$, $\left[n + h - \frac{1}{2}\right] = n - 1$.

$$\lim_{x \to n^+} f(x) = n \cos((n-1)\pi) = n(-1)^{n-1}$$

- $\mathbf{x \to n^-}$: $x = n - h$, $h \to 0^+$.

$$\lim_{x \to n^-} f(x) = \lim_{h \to 0^+} [n - h] \cos\left(\left[n - h - \frac{1}{2}\right]\pi\right)$$

Since $h > 0$, $[n - h] = n - 1$. Since $-h - \frac{1}{2} \in (-1, 0)$, $\left[n - h - \frac{1}{2}\right] = n - 1$.

$$\lim_{x \to n^-} f(x) = (n-1)\cos((n-1)\pi) = (n-1)(-1)^{n-1}$$

For continuity at $x = n$, the LHL must equal the RHL:

$$n(-1)^{n-1} = (n-1)(-1)^{n-1}$$

If $\cos((n-1)\pi) \neq 0$, we can divide by $(-1)^{n-1}$:

$$n = n - 1 \implies 0 = -1$$

This is impossible, so the LHL $\neq$ RHL for all integers n.

The function is discontinuous at **all integral points**.

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12. $\lim_{n\to\infty} \sin[\pi\sqrt{n^2+1}]$ is equal to:

   (a) ∞

   (b) **0**

   (c) does not exist

   (d) -1

   Solution: We want to evaluate $L = \lim_{n\to\infty} \sin[\pi\sqrt{n^2+1}]$. We rewrite $\sqrt{n^2+1}$ by multiplying and dividing by the conjugate:

$$\sqrt{n^2+1} - n = (\sqrt{n^2+1} - n)\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{(n^2+1) - n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n}$$

   Thus, $\sqrt{n^2+1} = n + \frac{1}{\sqrt{n^2+1}+n}$.

The argument of the sine function is:

$$\pi\sqrt{n^2+1} = \pi n + \frac{\pi}{\sqrt{n^2+1}+n}$$

Let $\theta_n = \frac{\pi}{\sqrt{n^2+1}+n}$. As $n \to \infty$, $\theta_n \to 0$.

$$L = \lim_{n\to\infty} \sin(\pi n + \theta_n)$$

Using the identity $\sin(A+B) = \sin A \cos B + \cos A \sin B$:

$$L = \lim_{n\to\infty} (\sin(\pi n)\cos(\theta_n) + \cos(\pi n)\sin(\theta_n))$$

Since n is an integer (implied by the sequence notation $n \to \infty$ for this kind of problem):

$$\sin(\pi n) = 0$$

$$\cos(\pi n) = (-1)^n$$

$$L = \lim_{n\to\infty} (0 \cdot \cos(\theta_n) + (-1)^n \sin(\theta_n)) = \lim_{n\to\infty} (-1)^n \sin(\theta_n)$$

Since $\theta_n \to 0$, we have $\sin(\theta_n) \to \sin(0) = 0$. Since $|(-1)^n \sin(\theta_n)| = |\sin(\theta_n)|$ and $\lim_{n\to\infty} |\sin(\theta_n)| = 0$, by the Squeeze Theorem, the limit is 0.

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13. The function defined by

$$f(x) = \begin{cases} |x-3|, & \text{if } x \geq 1 \\ \frac{1}{4}x^2 - \frac{3}{2}x + \frac{13}{4}, & \text{if } x < 1 \end{cases}$$

(a) **continuous at $x = 1$**

(b) **continuous at $x = 3$**

(c) differentiable at $x = 1$

(d) all of the above

Solution: We check continuity at the boundary points $x = 1$ and $x = 3$ (where the definition of $|x-3|$ changes).

At $x = 1$ (Boundary 1):

- LHL: $\lim_{x\to1^-} f(x) = \frac{1}{4}(1)^2 - \frac{3}{2}(1) + \frac{13}{4} = \frac{1}{4} - \frac{6}{4} + \frac{13}{4} = \frac{1+13-6}{4} = \frac{8}{4} = 2$
- RHL: $\lim_{x\to1^+} f(x) = |1-3| = |-2| = 2$
- $f(1) = |1-3| = 2$

Since LHL = RHL = $f(1)$, $f(x)$ is continuous at $x = 1$. (Option a is correct.)

At $x = 3$ (Boundary 2): Near $x = 3$, $x \geq 1$, so $f(x) = |x-3|$.

$$f(x) = \begin{cases} -(x-3) = 3-x, & \text{if } 1 \leq x < 3 \\ x-3, & \text{if } x \geq 3 \end{cases}$$

- LHL: $\lim_{x\to 3^-} f(x) = 3 - 3 = 0$
- RHL: $\lim_{x\to 3^+} f(x) = 3 - 3 = 0$
- $f(3) = 3 - 3 = 0$

Since LHL = RHL = $f(3)$, $f(x)$ is continuous at $x = 3$. (Option b is correct.)

Differentiability at $x = 1$: We find the left and right derivatives.

$$f'(x^-) = \frac{d}{dx}\left(\frac{1}{4}x^2 - \frac{3}{2}x + \frac{13}{4}\right) = \frac{1}{2}x - \frac{3}{2}$$

$$f'(1^-) = \frac{1}{2}(1) - \frac{3}{2} = -\frac{2}{2} = -1$$

Near $x = 1$ (for $x > 1$), $f(x) = |x - 3| = 3 - x$.

$$f'(x^+) = \frac{d}{dx}(3 - x) = -1$$

$$f'(1^+) = -1$$

Since $f'(1^-) = f'(1^+) = -1$, the function is differentiable at $x = 1$. (Option c is correct.)

Since a, b, and c are all correct, the answer is **d**.

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14. If

$$f(x) = \begin{cases} e^x, & \text{if } x \le 0 \\ |1 - x|, & \text{if } x > 0 \end{cases}$$

(a) $f(x)$ is differentiable at $x = 0$

(b) $f(x)$ **is continuous at** $x = 0, 1$

(c) $f(x)$ is differentiable at $x = 1$

(d) none of the above

Solution: We analyze continuity and differentiability at the boundary points $x = 0$ and $x = 1$.

At $x = 0$ (Boundary 1):

- LHL: $\lim_{x\to 0^-} f(x) = e^0 = 1$
- RHL: $\lim_{x\to 0^+} f(x) = |1 - 0| = 1$
- $f(0) = e^0 = 1$

Since LHL = RHL = $f(0)$, $f(x)$ is continuous at $x = 0$.

Differentiability at $x = 0$:

- Left Derivative ($x \le 0$): $f'(x^-) = \frac{d}{dx}(e^x) = e^x$. $f'(0^-) = e^0 = 1$.
- Right Derivative ($x > 0$): Near $x = 0$, $|1 - x| = 1 - x$. $f'(x^+) = \frac{d}{dx}(1 - x) = -1$. $f'(0^+) = -1$.

Since $f'(0^-) \neq f'(0^+)$, $f(x)$ is not differentiable at $x = 0$. (Option a is incorrect.)

At $x = 1$ (Boundary 2 for $|1-x|$): The function is $f(x) = |1-x|$ near $x = 1$.

$$f(x) = \begin{cases} 1-x, & \text{if } 0 < x \leq 1 \\ x-1, & \text{if } x > 1 \end{cases}$$

- LHL: $\lim_{x \to 1^-} f(x) = 1 - 1 = 0$
- RHL: $\lim_{x \to 1^+} f(x) = 1 - 1 = 0$
- $f(1) = 1 - 1 = 0$

Since LHL = RHL = $f(1)$, $f(x)$ is continuous at $x = 1$.

Differentiability at $x = 1$:

- Left Derivative ($0 < x \leq 1$): $f'(x^-) = \frac{d}{dx}(1-x) = -1$. $f'(1^-) = -1$.
- Right Derivative ($x > 1$): $f'(x^+) = \frac{d}{dx}(x-1) = 1$. $f'(1^+) = 1$.

Since $f'(1^-) \neq f'(1^+)$, $f(x)$ is not differentiable at $x = 1$. (Option c is incorrect.)

Since $f(x)$ is continuous at $x = 0$ and $x = 1$, Option **b** is correct.