SOLUTIONS FOR SET 4

1. **Question:** If $a, b, c \in \{1, 2, 3, 4, 5\}$, the number of equations of the form $ax^2 + bx + c = 0$ which have real roots is

   Solution: The equation $ax^2 + bx + c = 0$ has real roots if the discriminant $D \geq 0$.

   $$D = b^2 - 4ac \geq 0 \implies b^2 \geq 4ac$$

   Since $a, b, c \in \{1, 2, 3, 4, 5\}$, the total number of possible equations is $5 \times 5 \times 5 = 125$. We count the triples (a, b, c) satisfying $b^2 \geq 4ac$. We categorize by the value of b:

   (i) **If** $b = 1$: $1^2 \geq 4ac \implies 4ac \leq 1$. Since $a, c \geq 1$, $4ac \geq 4$. Thus, no solution for $b = 1$. (0 pairs)

   (ii) **If** $b = 2$: $2^2 \geq 4ac \implies 4ac \leq 4 \implies ac \leq 1$. Since $a, c \geq 1$, the only solution is $ac = 1$, which gives $(a, c) = (1, 1)$. (1 pair)

   (iii) **If** $b = 3$: $3^2 \geq 4ac \implies 4ac \leq 9 \implies ac \leq 2.25$. Since a, c are integers, $ac = 1$ or $ac = 2$.
   - $ac = 1$: $(1, 1)$
   - $ac = 2$: $(1, 2), (2, 1)$

   (3 pairs)

   (iv) **If** $b = 4$: $4^2 \geq 4ac \implies 16 \geq 4ac \implies ac \leq 4$.
   - $ac = 1$: $(1, 1)$
   - $ac = 2$: $(1, 2), (2, 1)$
   - $ac = 3$: $(1, 3), (3, 1)$
   - $ac = 4$: $(1, 4), (4, 1), (2, 2)$

   (7 pairs)

   (v) **If** $b = 5$: $5^2 \geq 4ac \implies 25 \geq 4ac \implies ac \leq 6.25$.
   - $ac = 1$: $(1, 1)$
   - $ac = 2$: $(1, 2), (2, 1)$
   - $ac = 3$: $(1, 3), (3, 1)$
   - $ac = 4$: $(1, 4), (4, 1), (2, 2)$
   - $ac = 5$: $(1, 5), (5, 1)$
   - $ac = 6$: $(2, 3), (3, 2)$

   (12 pairs)

   The total number of equations with real roots is the sum of pairs:

   $$\text{Total} = 0 + 1 + 3 + 7 + 12 = \mathbf{23}$$

   Re-evaluation of Options: The calculated answer is 23, which is not in the options. Let's assume the question meant $a, b, c \in \{1, 2, 3, 4, 5, 6\}$ (for example). If we strictly adhere to the set $\{1, 2, 3, 4, 5\}$, the answer is 23. Given the context of multiple-choice questions, and that 23 is close to 24, there might be a minor error in the question or options. Assuming the question and our calculation are correct, none of the options is correct. If forced to choose the closest option, it would be 24.

   Answer: None of the options matches the calculated value 23. Assuming there is a typo and the intended answer is **24**. (We will provide the answer for 24, as it is the closest option, likely due to a minor variation in the set not explicitly stated, but the mathematically derived answer is 23).

2. **Question:** In the equation $x^3 + 3Hx + G = 0$, if G and H are real and $G^2 + 4H^3 > 0$ then the roots are

   Solution: The equation $x^3 + 3Hx + G = 0$ is the reduced form of a general cubic equation. The nature of its roots is determined by the discriminant, Δ, which is related to G and H.

   (i) The discriminant of the reduced cubic equation $x^3 + 3Hx + G = 0$ is typically defined as:

   $$\Delta = -4(3H)^3 - 27G^2 = -108H^3 - 27G^2 = -27(4H^3 + G^2)$$

   (ii) We are given $G^2 + 4H^3 > 0$.

   (iii) Substitute this condition into the discriminant:

   $$\Delta = -27(G^2 + 4H^3)$$

   Since $G^2 + 4H^3 > 0$, we have $\Delta < 0$.

   (iv) According to the criteria for the nature of the roots of a cubic equation:

- If $\Delta > 0$, all three roots are real and distinct.
- If $\Delta = 0$, all roots are real, and at least two are equal.
- If $\Delta < 0$, **one root is real and two are non-real complex conjugates (imaginary)**.

(v) Since $\Delta < 0$, the roots are one real and two imaginary.

Answer: (c) one real and two imaginary

3. **Question:** If $[x]$ denotes the integral part of x and $k = \sin^{-1}\left(\frac{1+t^2}{2t}\right) > 0$ then the integral value of α for which the equation $(x-[k])(x+\alpha)-1=0$ has integral roots is

Solution:

(i) **Determine the value of k:** The domain of $\sin^{-1}(y)$ is $[-1, 1]$. We must have:

$$-1 \leq \frac{1+t^2}{2t} \leq 1$$

The inequality $\frac{1+t^2}{2t} \geq -1$ is equivalent to $1+t^2+2t \geq 0$ (for $t > 0$) or $1+t^2+2t \leq 0$ (for $t < 0$). $1+t^2+2t = (1+t)^2 \geq 0$, so this is always true. The inequality $\frac{1+t^2}{2t} \leq 1$ is equivalent to $1+t^2 \leq 2t \implies t^2-2t+1 \leq 0 \implies (t-1)^2 \leq 0$. Since $(t-1)^2 \geq 0$, the only possibility is $(t-1)^2 = 0$, which means $\mathbf{t = 1}$.

(ii) Substitute $t = 1$ into the expression for k:

$$k = \sin^{-1}\left(\frac{1+1^2}{2(1)}\right) = \sin^{-1}(1) = \frac{\pi}{2}$$

The condition $k > 0$ is satisfied ($\pi/2 > 0$).

(iii) **Determine the value of $[k]$:**

$$[k] = \left[\frac{\pi}{2}\right] \approx [1.5708] = \mathbf{1}$$

(iv) **Analyze the equation:** The equation is $(x-[k])(x+\alpha)-1=0$. Substitute $[k] = 1$:

$$(x-1)(x+\alpha)-1=0$$

$$x^2+(\alpha-1)x-\alpha-1=0$$

(v) **Integral Roots Condition:** Let the integral roots be r_1 and r_2.

- Sum of roots: $r_1+r_2 = -(\alpha-1) = 1-\alpha$
- Product of roots: $r_1r_2 = -(\alpha+1)$

From the sum, $\alpha = 1-r_1-r_2$. Substitute this into the product:

$$r_1r_2 = -((1-r_1-r_2)+1)$$

$$r_1r_2 = -(2-r_1-r_2) = r_1+r_2-2$$

$$r_1r_2-r_1-r_2+2=0$$

Complete the factorization (Simon's Favorite Factoring Trick) by adding $1-2=-1$ to both sides:

$$r_1r_2-r_1-r_2+1=-1$$

$$(r_1-1)(r_2-1)=-1$$

(vi) **Find the possible roots r_1, r_2:** Since r_1, r_2 are integers, r_1-1 and r_2-1 must be integer factors of -1.

- Case 1: $r_1-1=1$ and $r_2-1=-1 \implies r_1=2, r_2=0$.
- Case 2: $r_1-1=-1$ and $r_2-1=1 \implies r_1=0, r_2=2$.

The integral roots are 0 and 2.

(vii) **Find the integral value of α:** Use $r_1+r_2=1-\alpha$:

$$2+0=1-\alpha \implies 2=1-\alpha \implies \alpha = \mathbf{-1}$$

Since the calculated value $\alpha = -1$ is not in the options, we select "none of these".

Answer: (d) none of these

4. **Question: True/False:** The solution of $\log_{\frac{1}{3}}(2^{x+2} - 4^x) \geq -2$ is $(-\infty, \infty)$

   Solution:

   (i) **Domain Restriction:** The argument of the logarithm must be positive:
   $$2^{x+2} - 4^x > 0$$
   $$2^x \cdot 2^2 - (2^2)^x > 0$$
   $$4 \cdot 2^x - (2^x)^2 > 0$$
   Let $y = 2^x > 0$.
   $$4y - y^2 > 0 \implies y(4 - y) > 0$$
   Since $y > 0$, we need $4 - y > 0 \implies y < 4$. Substitute back $y = 2^x$: $2^x < 4 \implies 2^x < 2^2 \implies x < 2$. The domain is $\mathbf{x < 2}$. The solution set cannot be $(-\infty, \infty)$.

   (ii) **Solve the inequality:** The base of the logarithm is $1/3$, which is less than 1. When removing the log, the inequality sign flips.
   $$\log_{\frac{1}{3}}(2^{x+2} - 4^x) \geq -2$$
   $$2^{x+2} - 4^x \leq \left(\frac{1}{3}\right)^{-2}$$
   $$4 \cdot 2^x - (2^x)^2 \leq 9$$
   Let $y = 2^x$:
   $$4y - y^2 \leq 9 \implies 0 \leq y^2 - 4y + 9$$

   (iii) **Analyze the quadratic $y^2 - 4y + 9$:** The discriminant is $D = (-4)^2 - 4(1)(9) = 16 - 36 = -20$. Since the leading coefficient is $1 > 0$ and $D < 0$, the quadratic $y^2 - 4y + 9$ is always positive for all real y. Thus, $y^2 - 4y + 9 > 0$ is always true for $y = 2^x$.

   (iv) **Final Solution Set:** The solution set is defined only by the domain restriction $x < 2$. The solution is $(-\infty, 2)$.

   (v) **Conclusion:** Since the solution set is $(-\infty, 2)$ and not $(-\infty, \infty)$, the statement is False.

   Answer: False

5. **Question: True/False:** If $(\sqrt{2})^x + (\sqrt{3})^x = (\sqrt{13})^{x/2}$ then number of values of x is 4.

   Solution:

   (i) **Simplify the equation:**
   $$(\sqrt{2})^x + (\sqrt{3})^x = (\sqrt{13})^{x/2}$$
   $$2^{x/2} + 3^{x/2} = (13^{1/2})^{x/2} = 13^{x/4}$$
   This seems wrong. $13^{x/4} = (\sqrt{13})^{x/2}$ is correct. The equation is:
   $$2^{x/2} + 3^{x/2} = 13^{x/4}$$
   $$\left(2^{1/2}\right)^x + \left(3^{1/2}\right)^x = \left(13^{1/4}\right)^x$$
   This is $A^x + B^x = C^x$. Let's use the provided structure:
   $$(\sqrt{2})^x + (\sqrt{3})^x = \left(\sqrt{13}^{1/2}\right)^x = \left(13^{1/4}\right)^x$$
   This looks like $\sqrt{\sqrt{13}} = \sqrt[4]{13}$.

   Let's assume the question intended $(\sqrt{\mathbf{2}})^{\mathbf{x}} + (\sqrt{\mathbf{3}})^{\mathbf{x}} = (\sqrt{\mathbf{5}})^{\mathbf{x}}$. (Since $2 + 3 = 5$, this is a common form of such problems). If $(\sqrt{2})^x + (\sqrt{3})^x = (\sqrt{13})^{x/2}$ is correct, then:
   $$2^{x/2} + 3^{x/2} = (13^{1/2})^{x/2} = 13^{x/4} \implies 2^{x/2} + 3^{x/2} = (13^{1/2})^{x/2}$$
   This is confusing. Let's assume the base is intended to be a single number raised to x, i.e., $(\sqrt{\mathbf{2}})^{\mathbf{x}} + (\sqrt{\mathbf{3}})^{\mathbf{x}} = (\sqrt{\mathbf{13/4}})^{\mathbf{x}}$. Let's analyze the given equation as written:
   $$\sqrt{2}^x + \sqrt{3}^x = \sqrt{13^{x/2}} = \left(13^{1/2}\right)^{x/2} = 13^{x/4}$$
   Let $y = x/4$:
   $$2^{2y} + 3^{2y} = 13^y$$
   $$4^y + 9^y = 13^y$$

(ii) **Solve $4^y + 9^y = 13^y$:** Divide by 13^y (since $13^y \neq 0$):

$$\left(\frac{4}{13}\right)^y + \left(\frac{9}{13}\right)^y = 1$$

Let $f(y) = \left(\frac{4}{13}\right)^y + \left(\frac{9}{13}\right)^y$.

- Since $0 < 4/13 < 1$ and $0 < 9/13 < 1$, both $\left(\frac{4}{13}\right)^y$ and $\left(\frac{9}{13}\right)^y$ are strictly decreasing functions.
- Thus, $f(y)$ is a strictly decreasing function.
- A strictly monotonic function can intersect a horizontal line $y = 1$ at most once.

(iii) **Find the solution for y:** Test $y = 1$: $\left(\frac{4}{13}\right)^1 + \left(\frac{9}{13}\right)^1 = \frac{4+9}{13} = 1$. Thus, $\mathbf{y = 1}$ is the unique solution.

(iv) **Solve for x:** Since $y = x/4$, we have $1 = x/4 \implies \mathbf{x = 4}$.

(v) **Conclusion:** There is only **one** value of x, which is $x = 4$. The statement that the number of values of x is 4 is False.

Answer: False

6. **Question: True/False:** If $0 < x < 1000$ and $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = \frac{31}{30}x$, the number of possible values of x is 30.

Solution:

(i) **Use the property $[y] = y - \{y\}$:**

$$\left(\frac{x}{2} - \left\{\frac{x}{2}\right\}\right) + \left(\frac{x}{3} - \left\{\frac{x}{3}\right\}\right) + \left(\frac{x}{5} - \left\{\frac{x}{5}\right\}\right) = \frac{31}{30}x$$

$$\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5}\right)x - \left(\left\{\frac{x}{2}\right\} + \left\{\frac{x}{3}\right\} + \left\{\frac{x}{5}\right\}\right) = \frac{31}{30}x$$

(ii) **Simplify the coefficient of x:**

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{15 + 10 + 6}{30} = \frac{31}{30}$$

(iii) **Substitute back into the equation:**

$$\frac{31}{30}x - \left(\left\{\frac{x}{2}\right\} + \left\{\frac{x}{3}\right\} + \left\{\frac{x}{5}\right\}\right) = \frac{31}{30}x$$

$$\left\{\frac{x}{2}\right\} + \left\{\frac{x}{3}\right\} + \left\{\frac{x}{5}\right\} = 0$$

(iv) **Find possible values of x:** Since $0 \leq \{y\} < 1$, for the sum of three fractional parts to be zero, each fractional part must be zero:

$$\left\{\frac{x}{2}\right\} = 0, \quad \left\{\frac{x}{3}\right\} = 0, \quad \left\{\frac{x}{5}\right\} = 0$$

This means $x/2$, $x/3$, and $x/5$ must all be integers.

(v) **Determine the nature of x:** x must be a multiple of 2, 3, and 5. Therefore, x must be a multiple of the Least Common Multiple (LCM) of $2, 3, 5$.

$$\text{LCM}(2, 3, 5) = 2 \times 3 \times 5 = 30$$

So, x must be of the form $x = 30k$, where k is an integer.

(vi) **Count the number of values of x in the given range:** We are given $0 < x < 1000$.

$$0 < 30k < 1000$$

$$0 < k < \frac{1000}{30} = 33.33...$$

Since k must be an integer, $k \in \{1, 2, 3, \ldots, 33\}$. The number of possible values of k (and thus x) is 33.

(vii) **Conclusion:** The number of possible values of x is 33, not 30. The statement is False.

Answer: False

7. **Question: True/False:** If $5^x + (2\sqrt{3})^{2x} \geq 13^x$ then the solution set is $(-\infty, 2]$

Solution:

(i) **Simplify the inequality:**

$$5^x + (2\sqrt{3})^{2x} \geq 13^x$$
$$5^x + ((2\sqrt{3})^2)^x \geq 13^x$$
$$5^x + (4 \cdot 3)^x \geq 13^x$$
$$\mathbf{5^x + 12^x \geq 13^x}$$

(ii) **Analyze the equation $5^x + 12^x = 13^x$:** Divide by 13^x:

$$\left(\frac{5}{13}\right)^x + \left(\frac{12}{13}\right)^x = 1$$

Let $f(x) = \left(\frac{5}{13}\right)^x + \left(\frac{12}{13}\right)^x$. Since $0 < 5/13 < 1$ and $0 < 12/13 < 1$, $f(x)$ is a strictly decreasing function.

(iii) **Find the solution to $f(x) = 1$:** Recall the Pythagorean triple $5^2 + 12^2 = 25 + 144 = 169 = 13^2$. Test $x = 2$:

$$\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = \frac{25}{169} + \frac{144}{169} = \frac{169}{169} = 1$$

Thus, $\mathbf{x = 2}$ is the unique solution to the equality.

(iv) **Solve the inequality $f(x) \geq 1$:** Since $f(x)$ is strictly decreasing, $f(x) \geq 1$ holds for all x where x is less than or equal to the value where $f(x) = 1$.

$$f(x) \geq 1 \implies x \leq 2$$

(v) **Conclusion:** The solution set is $(-\infty, 2]$. The statement is True.

Answer: True

8. **Question: True/False:** If $a, b, c \in \mathbb{R}$ and $a + b + c = 0$, then the quadratic equation $4ax^2 + 3bx + 2c = 0$ has real roots.

Solution:

(i) For the quadratic equation $4ax^2 + 3bx + 2c = 0$ to have real roots, the discriminant D must be non-negative.

$$D = (3b)^2 - 4(4a)(2c) = 9b^2 - 32ac$$

(ii) We are given the relation $a + b + c = 0$, so $b = -a - c$.

(iii) Substitute b into the discriminant:

$$D = 9(-a - c)^2 - 32ac = 9(a^2 + 2ac + c^2) - 32ac$$
$$D = 9a^2 + 18ac + 9c^2 - 32ac = 9a^2 - 14ac + 9c^2$$

(iv) **Determine the sign of D:** Consider D as a quadratic in a. $D = 9a^2 - 14ac + 9c^2$.

- If $c = 0$: $b = -a$. $D = 9a^2$. Since $a \in \mathbb{R}$, $D \geq 0$.
- If $c \neq 0$: We can consider the discriminant of D itself, with respect to a.

$$D_a = (-14c)^2 - 4(9)(9c^2) = 196c^2 - 324c^2 = -128c^2$$

Since $c \neq 0$, $D_a < 0$. Since the leading coefficient of D (in terms of a) is $9 > 0$ and its discriminant $D_a < 0$, D is always positive (or zero if $a = c = 0$, which implies $a = b = c = 0$, trivial case).

(v) Thus, $D = 9a^2 - 14ac + 9c^2 \geq 0$ for all real a, c.

(vi) **Conclusion:** Since the discriminant $D \geq 0$, the quadratic equation always has real roots. The statement is True.

Answer: True

9. **Question: Fill in the blank:** The polynomial $x^3 - 3x^2 - 9x + c$ can be written as $(x - \alpha)^2(x - \beta)$ if $c =$ _____

Solution:

(i) If $P(x) = x^3 - 3x^2 - 9x + c$ can be written as $(x - \alpha)^2(x - \beta)$, then α is a root of multiplicity 2. This means that $P(\alpha) = 0$ and $P'(\alpha) = 0$.

(ii) The derivative is:

$$P'(x) = 3x^2 - 6x - 9$$

(iii) Find the roots of $P'(x) = 0$:

$$3x^2 - 6x - 9 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$

The possible values for the repeated root α are $\alpha = \mathbf{3}$ or $\alpha = \mathbf{-1}$.

(iv) **Case 1:** $\alpha = 3$ $P(3) = (3)^3 - 3(3)^2 - 9(3) + c = 0$

$$27 - 27 - 27 + c = 0 \implies \mathbf{c = 27}$$

(v) **Case 2:** $\alpha = -1$ $P(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + c = 0$

$$-1 - 3(1) + 9 + c = 0$$
$$-1 - 3 + 9 + c = 0 \implies 5 + c = 0 \implies \mathbf{c = -5}$$

(vi) The polynomial can be written in the given form if $c = 27$ or $c = -5$. We fill in one of the possible values.

Answer: 27 (or -5)

10. **Question: Fill in the blank:** If α, β are roots of $x^2 + px - q = 0$ and γ, δ are roots of $x^2 + px + r = 0$ then $(\alpha - \gamma)(\alpha - \delta) =$ -----

Solution:

(i) Let $f(x) = x^2 + px + r$. Since γ and δ are the roots of $f(x) = 0$, we can write $f(x)$ in factored form:

$$x^2 + px + r = (x - \gamma)(x - \delta)$$

(ii) The expression to be evaluated is $(\alpha - \gamma)(\alpha - \delta)$.

(iii) Substitute $x = \alpha$ into the factored form of $f(x)$:

$$(\alpha - \gamma)(\alpha - \delta) = \alpha^2 + p\alpha + r$$

(iv) Since α is a root of $x^2 + px - q = 0$:

$$\alpha^2 + p\alpha - q = 0 \implies \boldsymbol{\alpha^2 + p\alpha = q}$$

(v) Substitute this into the result from (iii):

$$(\alpha - \gamma)(\alpha - \delta) = (\alpha^2 + p\alpha) + r = q + r$$

Answer: $q + r$

11. **Question: Fill in the blank:** If $y \neq 0$ then the number of values of the pair (x, y) such that $x + y + \frac{x}{y} = \frac{1}{2}$ and $(x+y)\frac{x}{y} = -\frac{1}{2}$ is -----

Solution:

(i) Let $u = x + y$ and $v = \frac{x}{y}$. The system of equations becomes:

$$u + v = \frac{1}{2} \quad \text{(Eq. 1)}$$
$$uv = -\frac{1}{2} \quad \text{(Eq. 2)}$$

(ii) u and v are the roots of the quadratic equation $t^2 - (u+v)t + uv = 0$.

$$t^2 - \frac{1}{2}t - \frac{1}{2} = 0$$
$$2t^2 - t - 1 = 0$$
$$(2t+1)(t-1) = 0$$

The roots are $t = 1$ and $t = -1/2$.

(iii) Two possible pairs for $(u, v) = (x + y, x/y)$:

- Case 1: $u = 1$ and $v = -1/2$

- Case 2: $u = -1/2$ and $v = 1$

(iv) **Case 1:** $x + y = 1$ **and** $x/y = -1/2$

$$x = 1 - y$$

$$\frac{1-y}{y} = -\frac{1}{2} \implies 2(1-y) = -y \implies 2 - 2y = -y \implies y = 2$$

Substitute $y = 2$ back: $x = 1 - 2 = -1$. The pair is $(\mathbf{x}, \mathbf{y}) = (-\mathbf{1}, \mathbf{2})$. (Valid since $y \neq 0$).

(v) **Case 2:** $x + y = -1/2$ **and** $x/y = 1$

$$x = y$$

$$x + x = -1/2 \implies 2x = -1/2 \implies x = -1/4$$

Since $x = y$, $y = -1/4$. The pair is $(\mathbf{x}, \mathbf{y}) = (-\mathbf{1/4}, -\mathbf{1/4})$. (Valid since $y \neq 0$).

(vi) The total number of solutions (pairs (x, y)) is 2.

Answer: 2

12. **Question: Fill in the blank:** The value of α for which the equation $x^2 - (\sin\alpha - 2)x - (1 + \sin\alpha) = 0$ has roots whose sum of squares is least is _____

Solution:

(i) Let the roots be α' and β'.

(ii) Sum of roots: $S = \alpha' + \beta' = \sin\alpha - 2$.

(iii) Product of roots: $P = \alpha'\beta' = -(1 + \sin\alpha)$.

(iv) The sum of the squares of the roots is $E = (\alpha')^2 + (\beta')^2$:

$$E = (\alpha' + \beta')^2 - 2\alpha'\beta' = S^2 - 2P$$

$$E = (\sin\alpha - 2)^2 - 2(-(1 + \sin\alpha))$$

$$E = (\sin^2\alpha - 4\sin\alpha + 4) + 2 + 2\sin\alpha$$

$$E = \sin^2\alpha - 2\sin\alpha + 6$$

(v) Let $t = \sin\alpha$. Since $\alpha \in \mathbb{R}$, $t \in [-1, 1]$. We want to find the value of t that minimizes the quadratic $f(t) = t^2 - 2t + 6$.

(vi) **Minimize $f(t)$:** The vertex of the parabola $f(t) = t^2 - 2t + 6$ occurs at $t = -\frac{-2}{2(1)} = 1$. Since the coefficient of t^2 is $1 > 0$, $t = 1$ gives the minimum value.

(vii) Since $t = 1$ is within the allowed range $[-1, 1]$, the minimum occurs when $t = \sin\alpha = 1$.

(viii) The value of α for which $\sin\alpha = 1$ is $\alpha = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$. The simplest answer is the principal value.

Answer: $\frac{\pi}{2}$ (or $\frac{\pi}{2} + 2n\pi$)

13. **Question: Fill in the blank:** If the equations $x^2 - 5x + 6 = 0$ and $x^2 + mx + 3 = 0$ have a common root, then $m =$ _____

Solution:

(i) **Find the roots of the first equation:**

$$x^2 - 5x + 6 = 0$$

$$(x - 2)(x - 3) = 0$$

The roots are $\mathbf{x} = \mathbf{2}$ and $\mathbf{x} = \mathbf{3}$.

(ii) **Case 1: Common root is** $x = 2$ Substitute $x = 2$ into the second equation $x^2 + mx + 3 = 0$:

$$(2)^2 + m(2) + 3 = 0$$

$$4 + 2m + 3 = 0 \implies 2m = -7 \implies \mathbf{m} = -\mathbf{7/2}$$

(iii) **Case 2: Common root is** $x = 3$ Substitute $x = 3$ into the second equation $x^2 + mx + 3 = 0$:

$$(3)^2 + m(3) + 3 = 0$$

$$9 + 3m + 3 = 0 \implies 3m = -12 \implies \mathbf{m} = -\mathbf{4}$$

(iv) The possible values for m are $-7/2$ and -4. We fill in one of the possible values.

Answer: $-7/2$ (or -4)

14. **Question: Multiple Correct Choice:** If $px^2 + qx + r = 0$ and $rx^2 + qx + p = 0$ have a common non-real root, then

Solution:

(i) Let α be the common non-real root. Since the coefficients p, q, r are real, the complex conjugate $\bar{\alpha}$ must also be a root of both equations.

(ii) Since both equations have two common roots (α and $\bar{\alpha}$), the equations must be identical (proportional).

$$\frac{p}{r} = \frac{q}{q} = \frac{r}{p}$$

(iii) From $\frac{q}{q}$, we require $q \neq 0$.

(iv) From $\frac{p}{r} = \frac{r}{p}$:

$$p^2 = r^2 \implies p^2 - r^2 = 0 \implies (p-r)(p+r) = 0$$

Thus, $\mathbf{p = r}$ or $\mathbf{p = -r}$. (Option (c) and (d) are true).

(v) Now we analyze the condition for non-real roots. The discriminant D must be negative:

$$D = q^2 - 4pr < 0$$

(vi) **If $p = r$ (Option (d)):**

$$q^2 - 4p^2 < 0 \implies q^2 < 4p^2 \implies |q| < 2|p|$$

This means $-2|p| < q < 2|p|$. (Option (a) is true, and since $p = r$, $-2|r| < q < 2|r|$ which implies Option (b) is true).

(vii) **If $p = -r$ (and $p \neq 0$):**

$$q^2 - 4p(-p) < 0 \implies q^2 + 4p^2 < 0$$

Since $q^2 \geq 0$ and $4p^2 > 0$, $q^2 + 4p^2 > 0$. This is a contradiction. Thus, $p = -r$ is only possible if $p = r = 0$, but then the original equations are linear $qx = 0$, $q \neq 0$, which has one real root $x = 0$, contradicting the assumption of non-real root.

(viii) **Conclusion:** The only possibility is $\mathbf{p = r}$ and $q^2 < 4p^2$.

Answer: (a), (b), (c), (d) (Since $p = r$ implies all conditions)

15. **Question: Multiple Correct Choice:** If exactly one root of $x^2 - (t-1)x + t(t+4) = 0$ lies between the roots of $x^2 - (t+3)x + t + 2 = 0$ then

Solution:

(i) Let $f(x) = x^2 - (t-1)x + t(t+4)$ and $g(x) = x^2 - (t+3)x + t + 2$.

(ii) Find the roots of $g(x) = 0$:

$$x^2 - (t+3)x + (t+2) = 0$$

$$x^2 - (t+2+1)x + (t+2) = 0$$

$$(x-1)(x-(t+2)) = 0$$

The roots of $g(x) = 0$ are $\mathbf{r_1 = 1}$ and $\mathbf{r_2 = t+2}$.

(iii) Let the roots of $f(x) = 0$ be α, β. The condition is that exactly one of the roots $\{1, t+2\}$ lies between α and β. The general condition for a number k to lie between the roots of $f(x) = 0$ is $f(k) < 0$ (since the leading coefficient of $f(x)$ is $1 > 0$).

(iv) The condition "exactly one root of $g(x) = 0$ lies between the roots of $f(x) = 0$" means that $f(1)$ and $f(t+2)$ must have opposite signs (i.e., $\mathbf{f(1) \cdot f(t+2) < 0}$).

(v) **Calculate $f(1)$ and $f(t+2)$:**

$$f(1) = 1 - (t-1)(1) + t(t+4) = 1 - t + 1 + t^2 + 4t = t^2 + 3t + 2 = (t+1)(t+2)$$

$$f(t+2) = (t+2)^2 - (t-1)(t+2) + t(t+4)$$

$$f(t+2) = (t+2)[(t+2) - (t-1)] + t^2 + 4t$$

$$f(t+2) = (t+2)(3) + t^2 + 4t = 3t + 6 + t^2 + 4t = t^2 + 7t + 6 = (t+1)(t+6)$$

(vi) **Apply the condition $f(1) \cdot f(t+2) < 0$:**

$$[(t+1)(t+2)] \cdot [(t+1)(t+6)] < 0$$

$$(t+1)^2(t+2)(t+6) < 0$$

(vii) **Solve the inequality:** Since $(t+1)^2 \geq 0$, for the product to be negative, we must have:

$$(t+2)(t+6) < 0 \quad \text{AND} \quad t+1 \neq 0$$

- $(t+2)(t+6) < 0 \implies -6 < t < -2$.
- $t+1 \neq 0 \implies t \neq -1$.

(viii) **The solution for t is: $\mathbf{t} \in (\mathbf{-6}, \mathbf{-2}) \setminus \{\mathbf{-1}\}$.**

(ix) Compare with options:

- (a) $t \in (-6, -3)$: This is a subset of the solution, so it is a correct choice.
- (c) $t \in [-3, 1)$: This interval contains $t = -1$, which is excluded, but also $t \in [-3, -2)$ is a valid part of the solution. However, since the solution is $(-6, -2)$, this is generally incorrect.

Assuming the multiple correct choice requires only one root to be between α, β which covers the case $t = -1$. If $t = -1$, $f(1) = 0$, $f(t+2) < 0$. This means one root of $f(x)$ is $x = 1$ and the other is between 1 and -1. The statement holds. If $t = -3$, the solution is $(-6, -2)$. Option (a) is the only fully correct one among the intervals.

Answer: (a) $t \in (-6, -3)$