SOLUTIONS FOR SET 3

1. **Question:** If α and β are roots of $y^2 + py + q = 0$ and also $y^{2n} + p^n y^n + q^n = 0$ and if $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ are the roots of the $y^n + 1 + (y+1)^n = 0$ then n must be

   Solution:

   (i) Since α and β are common roots of $y^2 + py + q = 0$ and $y^{2n} + p^n y^n + q^n = 0$, as shown in a previous problem (Set 2, Q10), α^n and β^n must be the roots of $z^2 + p^n z + q^n = 0$. Thus, $\alpha^n + \beta^n = -p^n$ and $\alpha^n \beta^n = q^n$.

   (ii) Let the roots of the third equation be $y_1 = \dfrac{\alpha}{\beta}$ and $y_2 = \dfrac{\beta}{\alpha}$. The equation is $y^n + 1 + (y+1)^n = 0$.

   (iii) Substitute $y_1 = \dfrac{\alpha}{\beta}$:

   $$\left(\frac{\alpha}{\beta}\right)^n + 1 + \left(\frac{\alpha}{\beta} + 1\right)^n = 0$$

   $$\frac{\alpha^n}{\beta^n} + 1 + \left(\frac{\alpha + \beta}{\beta}\right)^n = 0$$

   Multiply by β^n:

   $$\alpha^n + \beta^n + (\alpha + \beta)^n = 0$$

   (iv) From $y^2 + py + q = 0$, we have $\alpha + \beta = -p$. Substitute this and the result from (i):

   $$(-p^n) + (-p)^n = 0$$

   $$(-p^n) + (-1)^n p^n = 0$$

   $$p^n[(-1) + (-1)^n] = 0$$

   (v) Since $p \neq 0$, we must have:

   $$-1 + (-1)^n = 0 \implies (-1)^n = 1$$

   This condition is only satisfied if n is an **even integer**.

   (vi) Check for the second root $y_2 = \dfrac{\beta}{\alpha}$:

   $$\left(\frac{\beta}{\alpha}\right)^n + 1 + \left(\frac{\beta}{\alpha} + 1\right)^n = 0$$

   Multiply by α^n:

   $$\beta^n + \alpha^n + (\beta + \alpha)^n = 0$$

   This leads to the same condition, n is an even integer.

   Answer: (c) an even integer

2. **Question:** Given a,b,c are real numbers. If α is a root of $a^2y^2 + by + c = 0$ and β is a root of $a^2y^2 - by - c = 0$ where $0 < \alpha < \beta$, then one root of $a^2y^2 + 2by + 2c = 0$ is γ such that

   Solution: Let $f_1(y) = a^2y^2 + by + c$ and $f_2(y) = a^2y^2 - by - c$. Let $g(y) = a^2y^2 + 2by + 2c$. We are looking for γ, a root of $g(y) = 0$.

   (i) Relate the polynomials: $g(y)$ can be expressed as a linear combination of $f_1(y)$ and $f_2(y)$:

   $$g(y) = 2f_1(y) - f_2(y) - (a^2y^2 + 2c)$$

   A simpler relation using α and β: Since α is a root of $f_1(y) = 0$:

   $$a^2\alpha^2 + b\alpha + c = 0 \implies b\alpha + c = -a^2\alpha^2$$

   Since β is a root of $f_2(y) = 0$:

   $$a^2\beta^2 - b\beta - c = 0 \implies b\beta + c = a^2\beta^2$$

   (ii) Evaluate $g(y)$ at $y = \alpha$ and $y = \beta$:

   $$g(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = a^2\alpha^2 + 2(b\alpha + c)$$

   Substitute $b\alpha + c = -a^2\alpha^2$:

   $$g(\alpha) = a^2\alpha^2 + 2(-a^2\alpha^2) = -a^2\alpha^2$$

   Since $a^2 > 0$ and $\alpha \neq 0$ (because $0 < \alpha < \beta$):

   $$\mathbf{g(\alpha) < 0}$$

(iii) Evaluate $g(\beta)$:

$$g(\beta) = a^2\beta^2 + 2b\beta + 2c = a^2\beta^2 + 2(b\beta + c)$$

From $b\beta + c = a^2\beta^2 - 2c$, no. Substitute $b\beta = a^2\beta^2 - c$.

$$g(\beta) = a^2\beta^2 + 2(a^2\beta^2 - c) + 2c = a^2\beta^2 + 2a^2\beta^2 - 2c + 2c = 3a^2\beta^2$$

Since $a^2 > 0$ and $\beta \neq 0$:

$$\mathbf{g(\beta) > 0}$$

(iv) Conclusion: Since $g(y)$ is a continuous function and $g(\alpha) < 0$ and $g(\beta) > 0$, by the Intermediate Value Theorem, there must be at least one root γ of $g(y) = 0$ between α and β.

$$\alpha < \gamma < \beta$$

(The equation $g(y) = 0$ is quadratic, so it has at most two roots. One root γ is guaranteed to be in (α, β)).

Answer: (d) $\alpha < \gamma < \beta$

3. **Question:** If α, β be the roots of $x^2 - px + q = 0$ and α', β' be the roots of $x^2 - p'x + q' = 0$ then the value of $(\alpha - \alpha')^2 + (\beta - \alpha')^2 + (\alpha - \beta')^2 + (\beta - \beta')^2$ is

Solution:

(i) From $x^2 - px + q = 0$:

$$\alpha + \beta = p \quad \text{and} \quad \alpha\beta = q$$
$$(\alpha + \beta)^2 = p^2 \implies \alpha^2 + \beta^2 = p^2 - 2q$$

(ii) From $x^2 - p'x + q' = 0$:

$$\alpha' + \beta' = p' \quad \text{and} \quad \alpha'\beta' = q'$$
$$(\alpha' + \beta')^2 = p'^2 \implies \alpha'^2 + \beta'^2 = p'^2 - 2q'$$

(iii) Let E be the expression:

$$E = (\alpha - \alpha')^2 + (\beta - \alpha')^2 + (\alpha - \beta')^2 + (\beta - \beta')^2$$

Expand the squared terms:

$$\begin{aligned} E &= (\alpha^2 - 2\alpha\alpha' + \alpha'^2) + (\beta^2 - 2\beta\alpha' + \alpha'^2) + (\alpha^2 - 2\alpha\beta' + \beta'^2) + (\beta^2 - 2\beta\beta' + \beta'^2) \\ &= 2(\alpha^2 + \beta^2) + 2(\alpha'^2 + \beta'^2) - 2\alpha'(\alpha + \beta) - 2\beta'(\alpha + \beta) \end{aligned}$$

(iv) Factor the last two terms:

$$E = 2(\alpha^2 + \beta^2) + 2(\alpha'^2 + \beta'^2) - 2(\alpha + \beta)(\alpha' + \beta')$$

(v) Substitute the values from (i) and (ii):

$$E = 2(p^2 - 2q) + 2(p'^2 - 2q') - 2(p)(p')$$
$$E = 2p^2 - 4q + 2p'^2 - 4q' - 2pp'$$
$$E = \mathbf{2\{p^2 - 2q + p'^2 - 2q' - pp'\}}$$

This matches option (a) when factoring out the 2.

Answer: (a) $2\{p^2 - 2q + p'^2 - 2q' - pp'\}$

4. **Question:** If the roots of the equation $(a - 1)(x^2 + x + 1)^2 = (a + 1)(x^4 + x^2 + 1)$ are real and distinct then the value of a $\in$

Solution:

(i) Use the identity: $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$.

(ii) The equation becomes:

$$(a - 1)(x^2 + x + 1)^2 = (a + 1)(x^2 + x + 1)(x^2 - x + 1)$$

(iii) Since $x^2 + x + 1 = (x + 1/2)^2 + 3/4 > 0$ for all real x, we can divide by $x^2 + x + 1$:

$$(a - 1)(x^2 + x + 1) = (a + 1)(x^2 - x + 1)$$

(iv) Expand and rearrange to form a quadratic equation $Ax^2 + Bx + C = 0$:

$$(a-1)x^2 + (a-1)x + (a-1) = (a+1)x^2 - (a+1)x + (a+1)$$
$$0 = [(a+1)-(a-1)]x^2 + [-(a+1)-(a-1)]x + [(a+1)-(a-1)]$$
$$0 = 2x^2 + (-2a)x + 2$$
$$x^2 - ax + 1 = 0$$

(v) The roots of this quadratic equation are real and distinct if the discriminant $D > 0$.

$$D = (-a)^2 - 4(1)(1) = a^2 - 4$$

(vi) Set $D > 0$:

$$a^2 - 4 > 0 \implies (a-2)(a+2) > 0$$

The inequality is satisfied when a is outside the roots of $a^2 - 4 = 0$:

$$\mathbf{a} \in (-\infty, -\mathbf{2}) \cup (\mathbf{2}, \infty)$$

(vii) We must also check the case where the factor $x^2 + x + 1 = 0$ was divided out. Since $x^2 + x + 1$ has no real roots, this division does not lose any real roots.

Answer: (b) $(-\infty, -2) \cup (2, \infty)$

5. **Question:** If the roots of the equation $ax^2 - bx + c = 0$ are α, β then the roots of the equation $b^2cx^2 - ab^2x + a^3 = 0$ are

Solution:

(i) From $ax^2 - bx + c = 0$:

$$\alpha + \beta = \frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

(ii) Consider the second equation $b^2cx^2 - ab^2x + a^3 = 0$. If x_0 is a root of this equation, then $b^2cx_0^2 - ab^2x_0 + a^3 = 0$.

(iii) We use the substitution method. Let the new root be y. The relation $x = g(y)$ is needed. Let's look at the reciprocals of the original roots: $1/\alpha, 1/\beta$. They satisfy $cx^2 - bx + a = 0$.

(iv) Rearrange the second equation: Divide by b^2c:

$$x^2 - \frac{a}{c}x + \frac{a^3}{b^2c} = 0$$

Using the relations $b/a = \alpha + \beta$ and $c/a = \alpha\beta$:

$$\frac{a}{c} = \frac{1}{\alpha\beta} \quad \text{and} \quad b^2 = a^2(\alpha+\beta)^2$$

The second equation is:

$$x^2 - \frac{1}{\alpha\beta}x + \frac{a^3}{a^2(\alpha+\beta)^2c} = 0$$
$$x^2 - \frac{1}{\alpha\beta}x + \frac{a}{c(\alpha+\beta)^2} = 0$$

Since $c/a = \alpha\beta$, $a/c = 1/\alpha\beta$:

$$x^2 - \frac{1}{\alpha\beta}x + \frac{1}{(\alpha\beta)^2(\alpha+\beta)^2}\frac{1}{\alpha\beta} = 0$$

This approach seems complicated.

(v) **Alternative approach (Testing options):** Let the new roots be y_1, y_2. Check if y_1 and y_2 satisfy the original equation $ax^2 - bx + c = 0$ if we substitute $x = h(y)$. Let the new roots be $y_1 = \dfrac{1}{\alpha^3 + \alpha\beta}$ and $y_2 = \dfrac{1}{\beta^3 + \alpha\beta}$. The term $\alpha^3 + \alpha\beta = \alpha(\alpha^2 + \beta)$. Since α is a root of $ax^2 - bx + c = 0$:

$$a\alpha^2 - b\alpha + c = 0 \implies \alpha^2 = \frac{b\alpha - c}{a}$$

(vi) Let's check the transformed root form. A common transformation is $y = \frac{1}{x^k}$. Let $x = 1/y$. Substitute $x = 1/y$ into the second equation: $b^2c(1/y)^2 - ab^2(1/y) + a^3 = 0$.

$$b^2c - ab^2y + a^3y^2 = 0 \implies a^3y^2 - ab^2y + b^2c = 0$$

This equation has roots $1/\alpha, 1/\beta$ if $a = b^2$ and $b = ab^2/a^3$ and $c = b^2c/a^3$. No.

Let's look at the roots of $a^3y^2 - ab^2y + b^2c = 0$. Sum of roots: $S = \frac{ab^2}{a^3} = \frac{b^2}{a^2}$. Product of roots: $P = \frac{b^2c}{a^3}$.

If the roots are $1/\alpha^2$ and $1/\beta^2$:

$$S = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = \frac{(b/a)^2 - 2c/a}{(c/a)^2} = \frac{b^2/a^2 - 2ac/a^2}{c^2/a^2} = \frac{b^2-2ac}{c^2}$$

This is not b^2/a^2.

The term in option (b) is $\frac{1}{\alpha^2+\alpha\beta}$ and $\frac{1}{\beta^2+\alpha\beta}$.

$$\alpha^2 + \alpha\beta = \alpha(\alpha+\beta) = \alpha(b/a)$$

$$\beta^2 + \alpha\beta = \beta(\beta+\alpha) = \beta(b/a)$$

New roots are $y_1 = \frac{1}{\alpha(b/a)} = \frac{a}{b\alpha}$ and $y_2 = \frac{a}{b\beta}$.

$$S' = \frac{a}{b}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{a}{b}\left(\frac{\alpha+\beta}{\alpha\beta}\right) = \frac{a}{b}\left(\frac{b/a}{c/a}\right) = \frac{a}{b}\cdot\frac{b}{c} = \frac{a}{c}$$

$$P' = \frac{a^2}{b^2\alpha\beta} = \frac{a^2}{b^2(c/a)} = \frac{a^3}{b^2c}$$

The new equation is $x^2 - S'x + P' = 0$:

$$x^2 - \frac{a}{c}x + \frac{a^3}{b^2c} = 0$$

Multiply by b^2c:

$$b^2cx^2 - ab^2x + a^3 = 0$$

This is exactly the second equation.

Answer: (b) $\frac{1}{\alpha^2+\alpha\beta}, \frac{1}{\beta^2+\alpha\beta}$

6. **Question:** If α, β are the roots of the equation $x^2 - ax + b = 0$ and $A_n = \alpha^n + \beta^n$, then which of the following is true?

Solution: This is a relationship based on the roots of a quadratic equation, known as a recurrence relation.

(i) Since α is a root of $x^2 - ax + b = 0$:

$$\alpha^2 - a\alpha + b = 0 \implies \alpha^2 = a\alpha - b$$

(ii) Multiply by α^{n-1}:

$$\alpha^{n+1} = a\alpha^n - b\alpha^{n-1}$$

(iii) Similarly, since β is a root:

$$\beta^{n+1} = a\beta^n - b\beta^{n-1}$$

(iv) Add the two equations:

$$\alpha^{n+1} + \beta^{n+1} = a(\alpha^n + \beta^n) - b(\alpha^{n-1} + \beta^{n-1})$$

(v) Substitute $A_k = \alpha^k + \beta^k$:

$$\mathbf{A_{n+1} = aA_n - bA_{n-1}}$$

Answer: (c) $A_{n+1} = aA_n - bA_{n-1}$

Question: Number of values of b for which equations $x^3 + bx + 1 = 0$ and $x^4 + bx^2 + 1 = 0$ have a common root

Solution: Let k be the common root.

$$k^3 + bk + 1 = 0 \tag{1}$$

$$k^4 + bk^2 + 1 = 0 \tag{2}$$

(i) Multiply equation (1) by k:

$$k(k^3 + bk + 1) = k^4 + bk^2 + k = 0$$

(ii) Substitute $k^4 + bk^2 = -1$ from equation (2) into the result from (i):

$$-1 + k = 0 \implies \mathbf{k = 1}$$

(iii) The only possible common root is $k = 1$. We must check if $k = 1$ satisfies both original equations for some value of b.

(iv) Substitute $k = 1$ into equation (1):

$$(1)^3 + b(1) + 1 = 0 \implies 1 + b + 1 = 0 \implies \mathbf{b = -2}$$

(v) Check $k = 1$ and $b = -2$ in equation (2):

$$(1)^4 + (-2)(1)^2 + 1 = 1 - 2 + 1 = 0$$

Since $k = 1$ satisfies both equations when $b = -2$, the equations have a common root when $b = -2$.

(vi) Thus, there is exactly **one** value of b for which the equations have a common root.

Answer: (b) 1

8. **Question:** Total number of integral values of a so that $x^2 - (a+1)x + a - 1 = 0$ has integral roots is equal to

Solution: Let α and β be the integral roots of $x^2 - (a+1)x + a - 1 = 0$.

(i) By Vieta's formulas:

$$\alpha + \beta = a + 1$$
$$\alpha\beta = a - 1$$

(ii) Eliminate a from the two equations:

$$a = \alpha + \beta - 1$$
$$\alpha\beta = (\alpha + \beta - 1) - 1$$
$$\alpha\beta = \alpha + \beta - 2$$

(iii) Rearrange the expression to factor:

$$\alpha\beta - \alpha - \beta + 2 = 0$$

Add 1 to both sides to complete the factoring pattern $\alpha\beta - \alpha - \beta + 1 = (\alpha - 1)(\beta - 1)$:

$$\alpha\beta - \alpha - \beta + 1 = -1$$
$$(\alpha - 1)(\beta - 1) = -1$$

(iv) Since α and β are integral roots, $\alpha - 1$ and $\beta - 1$ must be integers. The only pairs of integers whose product is -1 are $(1, -1)$ and $(-1, 1)$.

(v) **Case 1:** $\alpha - 1 = 1$ and $\beta - 1 = -1$.

$$\alpha = 2 \quad \text{and} \quad \beta = 0$$

Calculate a using $a = \alpha + \beta - 1$:

$$a = 2 + 0 - 1 = \mathbf{1}$$

(vi) **Case 2:** $\alpha - 1 = -1$ and $\beta - 1 = 1$.

$$\alpha = 0 \quad \text{and} \quad \beta = 2$$

Calculate a using $a = \alpha + \beta - 1$:

$$a = 0 + 2 - 1 = \mathbf{1}$$

(vii) There is only **one** distinct integral value of a, which is $a = 1$.

Answer: (a) 1

9. **Question:** If the equation $x^2 + ax + b = 0$ has distinct real roots and $x^2 + a|x| + b = 0$ has only one real root, then which of the following is true.

Solution: Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + a|x| + b$.

(i) $f(x) = 0$ has **distinct real roots** $\implies D = a^2 - 4b > 0$.

(ii) $g(x) = 0$ has **only one real root**. Since $g(x)$ is an even function, if $x = r$ is a root, then $x = -r$ is also a root (unless $r = 0$). If $r \neq 0$ is a root, then $g(x)$ has at least two roots ($\pm r$), which contradicts having only one real root. Therefore, the only possible real root must be $\mathbf{x = 0}$.

(iii) Substitute $x = 0$ into $g(x) = 0$:

$$(0)^2 + a|0| + b = 0 \implies \mathbf{b = 0}$$

(iv) Since $b = 0$, the first equation becomes $x^2 + ax = 0 \implies x(x + a) = 0$. The roots are $x = 0$ and $x = -a$.

(v) $f(x) = 0$ must have distinct real roots.

- If $a = 0$, the roots are $x = 0, 0$ (not distinct).
- If $a \neq 0$, the roots 0 and $-a$ are distinct.

So, we must have $\mathbf{a \neq 0}$.

(vi) Check the second equation with $b = 0$: $x^2 + a|x| = 0 \implies |x|(|x| + a) = 0$. The solutions are:

- $|x| = 0 \implies x = 0$ (one root).
- $|x| + a = 0 \implies |x| = -a$.

For $|x| = -a$ to have solutions, we must have $-a \geq 0$, or $a \leq 0$.

(vii) If $a < 0$, then $|x| = -a$ has two roots $x = \pm(-a) = \mp a$, which are $\neq 0$. This would mean $g(x) = 0$ has three roots $(0, -a, a)$, contradicting "only one real root".

(viii) Therefore, the only way $g(x) = 0$ has only one real root ($x = 0$) is if $|x| + a = 0$ has NO solutions for $x \neq 0$. This requires $|x| = -a$ to have no positive solutions for $|x|$.

$$|x| = -a$$

If $-a > 0 \implies a < 0$, then $|x| = -a$ has two distinct non-zero roots, $\pm(-a)$. If $-a = 0 \implies a = 0$, then $|x| = 0$, root is $x = 0$. If $-a < 0 \implies a > 0$, then $|x| = -a$ has no real roots.

(ix) Combining the required conditions:

- $b = 0$.
- $a \neq 0$ (for $f(x) = 0$ to have distinct roots).
- $a > 0$ (for $g(x) = 0$ to have only one root $x = 0$).

(x) The combined condition is $\mathbf{b = 0, a > 0}$.

Answer: (a) $b = 0, a > 0$

10. **Question:** If the equation $|x^2 + bx + c| = k$ has four real roots then

Solution: Let $f(x) = x^2 + bx + c$. The equation $|f(x)| = k$ has four distinct real roots if the graph of $y = |f(x)|$ intersects the horizontal line $y = k$ at four distinct points.

(i) $f(x) = x^2 + bx + c$ is a parabola opening upwards. Its vertex is at $x_v = -b/2$.

(ii) The minimum value of $f(x)$ is $f(x_v) = (-\frac{b}{2})^2 + b(-\frac{b}{2}) + c = \frac{b^2}{4} - \frac{b^2}{2} + c = c - \frac{b^2}{4}$. Let $m = f(x_v) = \frac{4c - b^2}{4}$.

(iii) For the graph of $y = |f(x)|$ to have a "V-shape" or "W-shape" near the vertex, the original parabola $f(x)$ must cross the x-axis, i.e., $f(x) = 0$ must have two distinct real roots.

$$D = b^2 - 4c > 0 \implies \mathbf{b^2 - 4c > 0}$$

This means the minimum value m must be negative: $m = c - b^2/4 < 0$. The graph dips below the x-axis.

(iv) The graph of $y = |f(x)|$ is obtained by reflecting the part of $f(x)$ below the x-axis. The maximum value of $|f(x)|$ near the vertex is $|m| = \left| c - \frac{b^2}{4} \right| = -\left(c - \frac{b^2}{4} \right) = \frac{b^2}{4} - c = \frac{b^2 - 4c}{4}$.

(v) For the line $y = k$ to intersect $y = |f(x)|$ at four distinct points, k must be positive and must be less than the height of the reflected "hump" below the x-axis.

$$0 < k < |m|$$

$$0 < k < \frac{b^2 - 4c}{4}$$

(vi) Recheck the options: Option (a) has $0 < k < \frac{4c - b^2}{4}$. Since $b^2 - 4c > 0$, we have $4c - b^2 < 0$. The upper bound for k cannot be negative. The correct upper bound is $\frac{b^2 - 4c}{4}$. The option (a) must have a typo and intended $\frac{b^2 - 4c}{4}$. We select (a) as the intended option but note the sign error.

Answer: (a) $b^2 - 4c > 0$ and $0 < k < \dfrac{4c - b^2}{4}$ (with the assumption that the upper bound should be $\dfrac{b^2 - 4c}{4}$)

11. **Question:** If a,b,c ,d $\in R$ then the equation $(x^2 + ax - 3b)(x^2 - cx + b)(x^2 - dx + 2b) = 0$ has

    Solution: The equation is $Q_1(x)Q_2(x)Q_3(x) = 0$. The overall equation has real roots if at least one of the quadratic factors has a non-negative discriminant.

    (i) Calculate the discriminants D_i for each quadratic factor:

    $$D_1 = a^2 - 4(1)(-3b) = a^2 + 12b$$

    $$D_2 = (-c)^2 - 4(1)(b) = c^2 - 4b$$

    $$D_3 = (-d)^2 - 4(1)(2b) = d^2 - 8b$$

    (ii) The equation has NO real roots if $D_1 < 0$ AND $D_2 < 0$ AND $D_3 < 0$.

    (iii) Assume, for contradiction, that there are no real roots (i.e., all $D_i < 0$).

    - $D_2 < 0 \implies c^2 < 4b$
    - $D_3 < 0 \implies d^2 < 8b$

    Both conditions imply that b must be positive, $\mathbf{b > 0}$.

    (iv) Now check D_1 with $b > 0$:

    $$D_1 = a^2 + 12b$$

    Since $a^2 \geq 0$ and $b > 0$, we have $D_1 = a^2 + 12b > 0$.

    (v) Contradiction: If $D_2 < 0$ and $D_3 < 0$, then D_1 must be positive.

    (vi) Therefore, it is impossible for all three discriminants to be simultaneously negative. Since $D_1 > 0$ for all a, c, d whenever $D_2 < 0$ and $D_3 < 0$, the first quadratic $Q_1(x)$ must have $\mathbf{D_1 > 0}$ if $b > 0$ and thus has two real roots. If $b \leq 0$, then $D_1 = a^2 + 12b$ may be ≥ 0 (if $a^2 \geq -12b$) or $D_3 = d^2 - 8b$ will be ≥ 0.

    (vii) If $b < 0$, $D_3 = d^2 - 8b > 0$, so $Q_3(x)$ has two real roots.

    (viii) In all cases, at least one discriminant is non-negative, meaning the overall equation has **at least two real roots**.

    Answer: (d) at least 2 real roots.

12. **Question:** Let α, β be the real and distinct roots of the equation $ax^2 + bx + c = |c|, (a > 0, c \neq 0)$ p, q be the real and distinct roots of the equation $ax^2 + bx + c = 0$. Then

    Solution: Let $f(x) = ax^2 + bx + c$.

    - The roots of $f(x) = |c|$ are α, β.
    - The roots of $f(x) = 0$ are p, q.

    The coefficient $a > 0$, so the parabola opens upward.

    (i) Since p, q are the roots of $f(x) = 0$, we have $f(p) = 0$ and $f(q) = 0$.

    (ii) α, β are the roots of $f(x) = |c|$. Assume $\alpha < \beta$.

    (iii) Since $c \neq 0$, $|c| > 0$. The roots p, q are the points where the parabola intersects the line $y = 0$. The roots α, β are the points where the parabola intersects the line $y = |c|$.

    (iv) Since the parabola opens upward ($a > 0$), and the line $y = |c|$ is above the line $y = 0$, the roots of $f(x) = |c|$ must be "further apart" than the roots of $f(x) = 0$.

    (v) Visualizing the graphs :

    $$f(\alpha) = |c| \quad \text{and} \quad f(\beta) = |c|$$

    Since $f(p) = 0$ and $f(q) = 0$ and $|c| > 0$, the value of $f(x)$ increases as x moves away from the axis of symmetry.

    (vi) Therefore, p and q must lie between α and β.

    $$\alpha < \mathbf{p} < \mathbf{q} < \beta \quad \text{or} \quad \alpha < \mathbf{q} < \mathbf{p} < \beta$$

    Answer: (a) p and q lie between α, β

13. **Question:** Let $f(x) = ax^2 + bx + c$ and $f(-1) < 1, f(1) > -1, f(3) < -4$, and $a \neq 0$, then

    Solution: We use the given inequalities to find the sign of a.

(i) Given inequalities:

$$\begin{aligned} f(-1) = a - b + c < 1 &\implies a - b + c - 1 < 0 \quad \text{(Eq. 1)} \\ f(1) = a + b + c > -1 &\implies a + b + c + 1 > 0 \quad \text{(Eq. 2)} \\ f(3) = 9a + 3b + c < -4 &\implies 9a + 3b + c + 4 < 0 \quad \text{(Eq. 3)} \end{aligned}$$

(ii) **Combine (Eq. 2) and (Eq. 1):** Subtracting (Eq. 1) from (Eq. 2):

$$(a + b + c) - (a - b + c) > -1 - 1$$

$$2b > -2 \implies b > -1$$

(iii) **Combine (Eq. 2) and (Eq. 3) to eliminate b:** Multiply (Eq. 2) by 3: $3(a+b+c) > -3 \implies 3a+3b+3c > -3$. Subtract this from (Eq. 3):

$$(9a + 3b + c) - (3a + 3b + 3c) < -4 - (-3)$$

$$6a - 2c < -1 \implies 2c > 6a + 1$$

(iv) **Combine (Eq. 1) and (Eq. 3) to eliminate b:** Multiply (Eq. 1) by 3: $3(a-b+c) < 3 \implies 3a-3b+3c < 3$. Add this to (Eq. 3):

$$(9a + 3b + c) + (3a - 3b + 3c) < -4 + 3$$

$$12a + 4c < -1$$

(v) Substitute $2c > 6a + 1$ from (iii) into $12a + 4c < -1$. $4c > 12a + 2$.

$$12a + (12a + 2) < -1$$

$$24a + 2 < -1$$

$$24a < -3$$

$$a < -\frac{3}{24} \implies \mathbf{a < -\frac{1}{8}}$$

(vi) Since $a < 0$, the parabola opens **downward**.

Answer: (b) a< 0

14. **Question:** A point (α, α^2) lies inside the triangle formed by the coordinate axes and the line $x + y = 6$. If α is a root of $f(x) = x^2 + ax + b = 0$ then which of the following is always true?

Solution:

(i) The triangle is defined by $x \geq 0$, $y \geq 0$, and $x + y \leq 6$.

(ii) The point (α, α^2) lies inside the triangle, so it must satisfy the boundary conditions strictly:

- $x > 0 \implies \alpha > \mathbf{0}$
- $y > 0 \implies \alpha^2 > 0$ (This is true since $\alpha > 0$)
- $x + y < 6 \implies \alpha + \boldsymbol{\alpha^2} < \mathbf{6}$

(iii) Solve the inequality $\alpha^2 + \alpha - 6 < 0$:

$$(\alpha + 3)(\alpha - 2) < 0$$

The roots are -3 and 2. For the inequality to be true, α must lie between the roots: $\mathbf{-3} < \alpha < \mathbf{2}$.

(iv) Combining the conditions from (ii) and (iii): $\mathbf{0} < \alpha < \mathbf{2}$.

(v) α is a root of $f(x) = x^2 + ax + b = 0$. The existence of this root $\alpha \in (0, 2)$ implies that the quadratic function $f(x)$ changes sign or touches the x-axis in the interval $(0, 2)$.

(vi) Evaluate the options using the knowledge that $f(\alpha) = 0$ for some $\alpha \in (0, 2)$:

- (a) $f(0) = b$. $f(0)$ could be positive or negative. For example, if roots are 0.5 and 1.5, $f(x) = (x-0.5)(x-1.5) = x^2 - 2x + 0.75$, $f(0) = 0.75 > 0$. If roots are $\alpha \in (0, 2)$ and $\beta > 2$, $f(0) = \alpha\beta > 0$. If roots are $\alpha \in (0, 2)$ and $\beta < 0$, $f(0) = \alpha\beta < 0$. $f(0) > 0$ is not always true.
- (b) $f(2) = 4 + 2a + b$. If α is the only root in $(0, 2)$, $f(x)$ must change sign at α. If $f(x)$ opens up (or down), $f(0)$ and $f(2)$ could have the same or opposite signs. Not always true.
- (c) $f(\beta) \leq 0$ for at least one $\beta \in (0, 2)$. Since α is a root and $\alpha \in (0, 2)$, $f(\alpha) = 0$. If we choose $\beta = \alpha$, then $f(\beta) = f(\alpha) = 0$. Since $0 \leq 0$, the statement $f(\beta) \leq 0$ for at least one $\beta \in (0, 2)$ is always true (by choosing $\beta = \alpha$).

(vii) The other options are not always true. For example, $f(0) = b = \alpha\beta$. If $\beta \in (0, 2)$, $f(0) > 0$. If β is a large negative number, $f(0) < 0$.

Answer: (c) $f(\beta) \leq 0$ for atleast one $\beta \in (0, 2)$