SOLUTIONS TO COMPLEX NUMBERS (SET 3)

1. If $w = \alpha + i\beta$, where $\beta \neq 0$, satisfies the condition that $\left(\frac{w-\overline{w}z}{1-z}\right)$ is purely real, then the set of values of z is : A. $|z| = 1, z \neq 2$ B. $|z| = 1$ and $z \neq 1$ C. $z = \overline{z}$ D. none of these

   Solution: Let $u = \frac{w-\overline{w}z}{1-z}$. Since u is purely real, $u = \overline{u}$.

$$\frac{w-\overline{w}z}{1-z} = \overline{\left(\frac{w-\overline{w}z}{1-z}\right)} = \frac{\overline{w}-w\overline{z}}{1-\overline{z}}$$

   Cross-multiply:

$$(w-\overline{w}z)(1-\overline{z}) = (\overline{w}-w\overline{z})(1-z)$$
$$w - w\overline{z} - \overline{w}z + \overline{w}z\overline{z} = \overline{w} - \overline{w}z - w\overline{z} + w\overline{z}z$$
$$w - w\overline{z} - \overline{w}z + \overline{w}|z|^2 = \overline{w} - \overline{w}z - w\overline{z} + w|z|^2$$

   Rearrange the terms:

$$w - \overline{w} = w|z|^2 - \overline{w}|z|^2$$
$$w - \overline{w} = |z|^2(w - \overline{w})$$

   We are given $w = \alpha + i\beta$ with $\beta \neq 0$.

$$w - \overline{w} = (\alpha + i\beta) - (\alpha - i\beta) = 2i\beta$$

   Since $\beta \neq 0$, $w - \overline{w} \neq 0$.

   We can divide by $w - \overline{w}$:

$$1 = |z|^2$$
$$|z| = 1$$

   Also, the denominator $1 - z$ must be non-zero, so $z \neq 1$.

   Answer: $|z| = 1$ and $z \neq 1$

2. A man walks a distance of 3 units from the origin towards the north east ($N45°E$) direction. From there, he walks a distance of 4 units towards the north west ($N45°W$) direction to reach a point P. Then the position of P in the Argand plane is : A. $3e^{\frac{i\pi}{4}} + 4i$ B. $(3-4i)e^{\frac{i\pi}{4}}$ C. $(4+3i)e^{\frac{i\pi}{4}}$ D. $(3+4i)e^{\frac{i\pi}{4}}$

   Solution: Let z_P be the complex number representing the position of P. The path is a sum of two displacement vectors.

   1. **First Displacement (z_1):** Distance $R_1 = 3$. Direction: North-East ($N45°E$), which is 45° from the positive real axis (East) in the positive direction (North).

$$\theta_1 = 45° = \frac{\pi}{4}$$
$$z_1 = 3e^{i\frac{\pi}{4}}$$

   2. **Second Displacement (z_2):** Distance $R_2 = 4$. Direction: North-West ($N45°W$). This direction is 45° from the positive imaginary axis (North) towards the negative real axis (West).

$$\theta_2 = 90° + 45° = 135° = \frac{3\pi}{4}$$
$$z_2 = 4e^{i\frac{3\pi}{4}}$$

The final position z_P is $z_1 + z_2$:

$$\begin{aligned}
z_P &= 3e^{i\frac{\pi}{4}} + 4e^{i\frac{3\pi}{4}} \\
&= 3e^{i\frac{\pi}{4}} + 4e^{i(\pi - \frac{\pi}{4})} \\
&= 3e^{i\frac{\pi}{4}} + 4\left(e^{i\pi}e^{-i\frac{\pi}{4}}\right) \\
&= 3e^{i\frac{\pi}{4}} + 4(-1)\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)
\end{aligned}$$

Let's use the first term $e^{i\frac{\pi}{4}}$ for factoring:

$$e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

Notice that $e^{i\frac{3\pi}{4}} = \frac{-1+i}{\sqrt{2}}$ and $e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$. Also, $ie^{i\frac{\pi}{4}} = i(\frac{1+i}{\sqrt{2}}) = \frac{i-1}{\sqrt{2}}$. No, this is not $e^{i\frac{3\pi}{4}}$. $e^{i\frac{3\pi}{4}} = e^{i\frac{\pi}{2}}e^{i\frac{\pi}{4}} = ie^{i\frac{\pi}{4}}$.

Let's check the argument: $\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$. This is correct.

$$\begin{aligned}
z_P &= 3e^{i\frac{\pi}{4}} + 4\left(ie^{i\frac{\pi}{4}}\right) \\
z_P &= e^{i\frac{\pi}{4}}(3+4i)
\end{aligned}$$

Answer: $(3+4i)e^{\frac{i\pi}{4}}$

3. If $|z| = 1$ and $z \neq \pm 1$ then the values of $w = \frac{z}{1-z^2}$ lie on : A. a line not passing through the origin B. $|z| = \sqrt{2}$ C. x axis D. y axis

   Solution: Given $|z| = 1$, we use $z\overline{z} = 1$, so $\overline{z} = \frac{1}{z}$.

   Consider the conjugate of w:

$$\begin{aligned}
\overline{w} &= \overline{\left(\frac{z}{1-z^2}\right)} \\
&= \frac{\overline{z}}{1-\overline{z}^2}
\end{aligned}$$

   Substitute $\overline{z} = \frac{1}{z}$:

$$\begin{aligned}
\overline{w} &= \frac{\frac{1}{z}}{1 - \left(\frac{1}{z}\right)^2} \\
&= \frac{\frac{1}{z}}{1 - \frac{1}{z^2}} \\
&= \frac{\frac{1}{z}}{\frac{z^2-1}{z^2}} \\
&= \frac{1}{z} \cdot \frac{z^2}{z^2-1} \\
&= \frac{z}{z^2-1} \\
&= -\frac{z}{1-z^2} \\
&= -w
\end{aligned}$$

   Since $\overline{w} = -w$, this means $\text{Re}(w) = 0$. If $w = u + iv$, then $\overline{w} = u - iv$.

$$u - iv = -(u+iv) = -u - iv$$

$$u = -u \implies 2u = 0 \implies u = 0$$

Since the real part is zero, w is purely imaginary. In the Argand plane, a purely imaginary number lies on the **imaginary axis** (y-axis).

The condition $z \neq \pm 1$ ensures that the denominator $1 - z^2 \neq 0$.

Answer: y axis

Fill in the Blanks

4. ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. If the points D and M represent the complex numbers $1 + i$ and $2 - i$ respectively, then A represents the complex number ...... or .....

Solution: In a rhombus, the diagonals bisect each other at M. M is the midpoint of AC and the midpoint of BD. $D(z_D) = 1 + i$ and $M(z_M) = 2 - i$. Let $A(z_A)$, $C(z_C)$ and $B(z_B)$ be the other vertices.

1. **Find B(z_B):** Since M is the midpoint of BD, $z_M = \frac{z_B + z_D}{2}$.

$$z_B = 2z_M - z_D = 2(2 - i) - (1 + i) = 4 - 2i - 1 - i = 3 - 3i$$

2. **Use the length condition:** The diagonals are perpendicular. $AC \perp BD$. $BD = 2|z_D - z_M| = 2|(1 + i) - (2 - i)| = 2| - 1 + 2i| = 2\sqrt{(-1)^2 + 2^2} = 2\sqrt{5}$.

Given $BD = 2AC$, we have $2\sqrt{5} = 2AC$, so $AC = \sqrt{5}$.

Since M is the midpoint of AC, $AM = MC = \frac{1}{2}AC = \frac{\sqrt{5}}{2}$.

3. **Vector relations:** A lies on the line perpendicular to BD at M. The vector $\vec{MD} = z_D - z_M = -1 + 2i$. The vector $\vec{MA} = z_A - z_M$. Since $\vec{MA} \perp \vec{MD}$, we have $z_A - z_M = \lambda i(z_D - z_M)$, where λ is a real scalar.

The magnitude condition is $|z_A - z_M| = AM = \frac{\sqrt{5}}{2}$.

$$|z_A - z_M| = |\lambda i(-1 + 2i)| = |\lambda i| \cdot | - 1 + 2i| = |\lambda| \cdot 1 \cdot \sqrt{5}$$

$$|\lambda|\sqrt{5} = \frac{\sqrt{5}}{2} \implies |\lambda| = \frac{1}{2} \implies \lambda = \pm\frac{1}{2}$$

Case 1: $\lambda = \frac{1}{2}$

$$z_A - z_M = \frac{1}{2}i(-1 + 2i) = \frac{1}{2}(-i + 2i^2) = \frac{1}{2}(-2 - i) = -1 - \frac{1}{2}i$$

$$z_A = z_M + (-1 - \frac{1}{2}i) = (2 - i) + (-1 - \frac{1}{2}i) = 1 - \frac{3}{2}i$$

Case 2: $\lambda = -\frac{1}{2}$

$$z_A - z_M = -\frac{1}{2}i(-1 + 2i) = -\left(-1 - \frac{1}{2}i\right) = 1 + \frac{1}{2}i$$

$$z_A = z_M + (1 + \frac{1}{2}i) = (2 - i) + (1 + \frac{1}{2}i) = 3 - \frac{1}{2}i$$

Answer: $1 - \frac{3}{2}i$ or $3 - \frac{1}{2}i$

5. Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then $z_2 =$ ...., $z_3 =$ .....

Solution: The vertices of an equilateral triangle inscribed in the circle $|z| = R$ are obtained by multiplying any vertex z_k by the cube roots of unity, ω and ω^2, provided the circumcenter (here, the origin 0) is also the centroid.

First, check if z_1 lies on the circle $|z| = 2$:

$$|z_1| = |1 + i\sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = 2$$

Since $|z_1| = 2$, the condition is satisfied.

The next two vertices z_2 and z_3 are obtained by rotating z_1 by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ about the origin.

$$z_2 = z_1\omega \quad \text{and} \quad z_3 = z_1\omega^2$$

where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Alternatively, write z_1 in polar form:

$$z_1 = 1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2e^{i\frac{\pi}{3}}$$

The other vertices are:

$$z_2 = 2e^{i(\frac{\pi}{3}+\frac{2\pi}{3})} = 2e^{i\pi} = 2(-1) = -2$$

$$z_3 = 2e^{i(\frac{\pi}{3}+\frac{4\pi}{3})} = 2e^{i\frac{5\pi}{3}} = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

$$\cos\frac{5\pi}{3} = \cos(-\frac{\pi}{3}) = \frac{1}{2}$$

$$\sin\frac{5\pi}{3} = \sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$$

$$z_3 = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$$

Answer: $z_2 = -2, z_3 = 1 - i\sqrt{3}$ (or vice versa)

6. The value of the expression $(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + ... + (n-1)(n-\omega)(n-\omega^2)$, where ω is an imaginary cube root of unity, is....

   Solution: We use the identity:

$$(k-\omega)(k-\omega^2) = k^2 - k(\omega+\omega^2) + \omega^3$$

Since $1 + \omega + \omega^2 = 0$, we have $\omega + \omega^2 = -1$, and $\omega^3 = 1$.

$$(k-\omega)(k-\omega^2) = k^2 - k(-1) + 1 = k^2 + k + 1$$

Let T_k be the k-th term of the series. The index k runs from 2 to n.

$$T_k = (k-1)(k-\omega)(k-\omega^2)$$

The term in the original sum corresponding to $k = 2$ is $(2-1)(2-\omega)(2-\omega^2) = 1(2^2+2+1) = 7$.

The term in the original sum corresponding to k is $(k-\omega)(k-\omega^2)$ if we assume the coefficient is 1. The general term provided has an increasing coefficient starting from $1 \cdot (2-\omega)(2-\omega^2)$ for $k = 2$.

The series is:

$$S = \sum_{k=2}^{n}(k-1)\cdot(k-\omega)(k-\omega^2)$$

Substitute the identity:

$$S = \sum_{k=2}^{n}(k-1)(k^2+k+1) = \sum_{k=2}^{n}(k^3-1)$$

The summation $\sum_{k=1}^{n}(k^3-1)$ is $\sum_{k=1}^{n}k^3 - \sum_{k=1}^{n}1$. The given sum starts at $k = 2$:

$$S = \left(\sum_{k=1}^{n}k^3 - \sum_{k=1}^{n}1\right) - (1^3 - 1)$$

$$S = \left(\sum_{k=1}^{n} k^3 - n \right) - 0$$

We use the formula for the sum of the first n cubes: $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

$$S = \left(\frac{n(n+1)}{2}\right)^2 - n$$

Answer: $\left(\frac{n(n+1)}{2}\right)^2 - n$

TRUE / FALSE

7. The cube roots of unity when represented on argand diagram form the vertices of an equilateral triangle.

   Solution: The cube roots of unity are $z_1 = 1, z_2 = \omega, z_3 = \omega^2$.

   - $|z_1 - z_2| = |1 - \omega|$
   - $|z_2 - z_3| = |\omega - \omega^2| = |\omega(1 - \omega)| = |\omega||1 - \omega| = 1 \cdot |1 - \omega| = |1 - \omega|$
   - $|z_3 - z_1| = |\omega^2 - 1| = |-(1-\omega^2)| = |1-\omega^2| = |(1-\omega)(1+\omega)| = |1-\omega| \cdot |-\omega^2| = |1-\omega| \cdot 1 = |1-\omega|$

   Since the magnitudes of all three side lengths are equal, the triangle is equilateral.

   Answer: TRUE

OBJECTIVE QUESTIONS More than one options are correct

8. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1\overline{z_2}) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies : A. $|w_1| = 1$ B. $|w_2| = 1$ C. $\text{Re}(w_1\overline{w_2}) = 0$ D. none of these

   Solution: 1. **Use $|z_1| = |z_2| = 1$**: $|z_1|^2 = a^2 + b^2 = 1$ $|z_2|^2 = c^2 + d^2 = 1$

   2. **Use $\text{Re}(z_1\overline{z_2}) = 0$**:

$$z_1\overline{z_2} = (a + ib)(c - id) = (ac + bd) + i(bc - ad)$$

$$\text{Re}(z_1\overline{z_2}) = ac + bd = 0 \implies ac = -bd$$

   3. **Check the options for $w_1 = a + ic$ and $w_2 = b + id$**:

   $|w_1| = 1$:

$$|w_1|^2 = a^2 + c^2$$

   From $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$:

$$a^2 + c^2 = a^2 + (1 - d^2) = (1 - b^2) + c^2$$

   Since a, b, c, d are related by $ac = -bd$, we must check if $a^2 + c^2 = 1$.

   From $ac = -bd$, square both sides: $a^2c^2 = b^2d^2$. Substitute $b^2 = 1 - a^2$ and $d^2 = 1 - c^2$:

$$a^2c^2 = (1 - a^2)(1 - c^2) = 1 - a^2 - c^2 + a^2c^2$$

$$0 = 1 - a^2 - c^2 \implies a^2 + c^2 = 1$$

   Thus, $|w_1|^2 = 1$, so $|w_1| = 1$. (Option A is correct).

   $|w_2| = 1$:

$$|w_2|^2 = b^2 + d^2$$

   From $a^2 + c^2 = 1$ (shown above), and $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$, we have:

$$b^2 + d^2 = (1 - a^2) + (1 - c^2) = 2 - (a^2 + c^2) = 2 - 1 = 1$$

Thus, $|w_2|^2 = 1$, so $|w_2| = 1$. (Option B is correct).

$\text{Re}(w_1\overline{w_2}) = 0$:

$$w_1\overline{w_2} = (a+ic)(b-id) = (ab+cd) + i(bc-ad)$$
$$\text{Re}(w_1\overline{w_2}) = ab+cd$$

We want to check if $ab+cd=0$. This is not guaranteed by $ac+bd=0$.

For example, let $z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ and $z_2 = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. $|z_1| = |z_2| = 1$. $z_1\overline{z_2} = (\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})^2 = \frac{1}{2} - \frac{1}{2} + i(1) = i$. $\text{Re}(z_1\overline{z_2}) = 0$. (Conditions satisfied). $a = 1/\sqrt{2}, b = 1/\sqrt{2}, c = 1/\sqrt{2}, d = -1/\sqrt{2}$. $w_1 = a+ic = \frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$ $w_2 = b+id = \frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}$ $\text{Re}(w_1\overline{w_2}) = ab+cd = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}(-\frac{1}{\sqrt{2}}) = \frac{1}{2}-\frac{1}{2} = 0$.

This specific example satisfies all three. Let's try to find a counterexample for C. Consider $z_1 = i, z_2 = 1$. $|z_1| = |z_2| = 1$. $\overline{z_2} = 1$. $z_1\overline{z_2} = i$. $\text{Re}(z_1\overline{z_2}) = 0$. (Conditions satisfied). $a = 0, b = 1, c = 1, d = 0$. $w_1 = a+ic = 0+i(1) = i$. $|w_1| = 1$. $w_2 = b+id = 1+i(0) = 1$. $|w_2| = 1$. $w_1\overline{w_2} = i(1) = i$. $\text{Re}(w_1\overline{w_2}) = 0$. (Option C is also correct).

In fact, the relation $a^2+c^2 = 1$ and $b^2+d^2 = 1$ implies that the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has orthogonal columns of unit length. Since the columns are orthogonal, the rows are also orthogonal and of unit length.

$$a^2+b^2 = 1 \quad c^2+d^2 = 1 \quad ac+bd = 0$$

This implies:

$$a^2+c^2 = 1 \quad b^2+d^2 = 1 \quad ab+cd = 0$$

The third condition $ab+cd=0$ is exactly $\text{Re}(w_1\overline{w_2}) = 0$.

Answer: $|w_1| = 1$, $|w_2| = 1$, $\text{Re}(w_1\overline{w_2}) = 0$

9. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$ if z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1+z_2}{z_1-z_2}$ may be : A. zero B. real and positive C. real and negative D. purely imaginary E. none of these

   Solution: Let $w = \frac{z_1+z_2}{z_1-z_2}$.

   Since $|z_1| = |z_2|$, z_1 and z_2 lie on a circle centered at the origin. The expression w is the ratio of the diagonal sums of the parallelogram formed by z_1, z_2 and the origin.

   We use the property:

$$w = \frac{z_1+z_2}{z_1-z_2} = \frac{(z_1+z_2)(\overline{z_1-z_2})}{(z_1-z_2)(\overline{z_1-z_2})} = \frac{z_1\overline{z_1} - z_1\overline{z_2} + z_2\overline{z_1} - z_2\overline{z_2}}{|z_1-z_2|^2}$$

   Since $|z_1| = |z_2| = R$, $|z_1|^2 = |z_2|^2 = R^2$.

$$w = \frac{R^2 - z_1\overline{z_2} + z_2\overline{z_1} - R^2}{|z_1-z_2|^2} = \frac{z_2\overline{z_1} - z_1\overline{z_2}}{|z_1-z_2|^2}$$

   Let $z_2\overline{z_1} = X+iY$. Then $z_1\overline{z_2} = \overline{z_2\overline{z_1}} = X-iY$.

$$w = \frac{(X+iY)-(X-iY)}{|z_1-z_2|^2} = \frac{2iY}{|z_1-z_2|^2}$$

   Since $|z_1-z_2|^2$ is a positive real number, w is **purely imaginary** (or zero if $Y=0$).

   Now we check the condition for $Y=0$: Y is the imaginary part of $z_2\overline{z_1}$.

$$z_2\overline{z_1} = |z_1||z_2|e^{i(\arg z_2 - \arg z_1)} = R^2e^{i(\theta_2-\theta_1)}$$

   $Y = R^2\sin(\theta_2-\theta_1)$. $Y = 0$ if $\sin(\theta_2-\theta_1) = 0$, which means $\theta_2-\theta_1 = k\pi$. Since $z_1 \neq z_2$, the difference cannot be 0 or 2π. The difference must be $\pm\pi$, which means $z_1 = -z_2$.

   Condition given: z_1: $\text{Re}(z_1) > 0$. z_1 is in the right half-plane. z_2: $\text{Im}(z_2) < 0$. z_2 is in the lower half-plane.

If $z_1 = -z_2$, then $\text{Re}(z_2) < 0$ and $\text{Im}(z_2)$ has the opposite sign of $\text{Im}(z_1)$. Let $z_1 = x + iy$. $x > 0$. $|z_1| = R$. $z_2 = -x - iy$. Since $\text{Im}(z_2) = -y < 0$, we must have $y > 0$. So z_1 is in Q1 and z_2 is in Q3. This satisfies both $|z_1| = |z_2|$ and the quadrant conditions.

In this case ($z_1 = -z_2$), the denominator $z_1 - z_2 = 2z_1 \neq 0$. The numerator $z_1 + z_2 = 0$.

$$w = \frac{0}{2z_1} = 0$$

So w may be **zero**. (Option A is correct).

If $z_1 \neq -z_2$, then $Y \neq 0$, and w is **purely imaginary**. (Option D is correct).

Answer: zero, purely imaginary

SUBJECTIVE QUESTIONS

10. It is given that n is an odd integer greater than 3, but n is not a multiple of 3. Prove that $x^3 + x^2 + x$ is a factor of $(x+1)^n - x^n - 1$.

Solution: Let $P(x) = (x+1)^n - x^n - 1$.

The factor is $x^3 + x^2 + x = x(x^2 + x + 1)$. For $x(x^2 + x + 1)$ to be a factor of $P(x)$, we must show that $P(x)$ has roots $0, \omega, \omega^2$, where ω is a cube root of unity ($\omega^3 = 1, \omega^2 + \omega + 1 = 0$).

1. **Check $x = 0$:**

$$P(0) = (0+1)^n - 0^n - 1 = 1^n - 0 - 1 = 1 - 1 = 0$$

Since $P(0) = 0$, x is a factor of $P(x)$.

2. **Check $x = \omega$:**

$$P(\omega) = (\omega + 1)^n - \omega^n - 1$$

Since $\omega^2 + \omega + 1 = 0$, $\omega + 1 = -\omega^2$.

$$P(\omega) = (-\omega^2)^n - \omega^n - 1$$

Since n is an odd integer, $(-1)^n = -1$.

$$P(\omega) = -(\omega^2)^n - \omega^n - 1 = -\omega^{2n} - \omega^n - 1$$

Since n is not a multiple of 3, n is of the form $3k+1$ or $3k+2$. Since n is odd, if $n = 3k+1$, k is even. If $n = 3k+2$, k is odd. $n \not\equiv 0 \pmod 3$.

Case 1: $n = 3k+1$. Then $\omega^n = \omega^{3k+1} = (\omega^3)^k \omega = \omega$.

$$\omega^{2n} = \omega^{6k+2} = (\omega^3)^{2k} \omega^2 = \omega^2$$

$$P(\omega) = -\omega^2 - \omega - 1 = -(\omega^2 + \omega + 1) = -(0) = 0$$

Case 2: $n = 3k+2$. Then $\omega^n = \omega^{3k+2} = \omega^2$.

$$\omega^{2n} = \omega^{6k+4} = \omega^4 = \omega$$

$$P(\omega) = -\omega - \omega^2 - 1 = -(\omega + \omega^2 + 1) = -(0) = 0$$

Since $P(\omega) = 0$, $x - \omega$ is a factor of $P(x)$.

3. **Check $x = \omega^2$:** Since the coefficients of $P(x)$ are real, if ω is a root, its conjugate $\overline{\omega} = \omega^2$ is also a root. $P(\omega^2) = \overline{P(\omega)} = \overline{0} = 0$. Since $P(\omega^2) = 0$, $x - \omega^2$ is a factor of $P(x)$.

Since $x, (x-\omega), (x-\omega^2)$ are factors of $P(x)$, their product $x(x-\omega)(x-\omega^2) = x(x^2 - (\omega+\omega^2)x + \omega^3) = x(x^2 - (-1)x + 1) = x(x^2 + x + 1) = x^3 + x^2 + x$ is a factor of $(x+1)^n - x^n - 1$.

11. Find the real values of x and y for which the following equation is satisfied : $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$

Solution: Multiply the entire equation by the common denominator $(3+i)(3-i) = 3^2 - i^2 = 9 + 1 = 10$:

$$(3-i)((1+i)x - 2i) + (3+i)((2-3i)y + i) = 10i$$

Expand the left side:

$$\begin{aligned}
\text{LHS} &= [(3-i)(1+i)x - 2i(3-i)] + [(3+i)(2-3i)y + i(3+i)] \\
&= [(3+3i-i-i^2)x - (6i-2i^2)] + [(6-9i+2i-3i^2)y + (3i+i^2)] \\
&= [(4+2i)x - (-2+6i)] + [(9-7i)y + (-1+3i)] \\
&= (4x+2ix+2-6i) + (9y-7iy-1+3i) \\
&= (4x+9y+1) + i(2x-7y-3)
\end{aligned}$$

Equate LHS to $10i$:

$$(4x+9y+1) + i(2x-7y-3) = 0 + 10i$$

Equating the real and imaginary parts gives a system of two linear equations:

1. Real part: $4x + 9y + 1 = 0 \implies 4x + 9y = -1 \quad (*)$ 2. Imaginary part: $2x - 7y - 3 = 10 \implies 2x - 7y = 13 \quad (**)$

Multiply $(**)$ by 2:

$$4x - 14y = 26 \quad (***)$$

Subtract $(***)$ from $(*)$:

$$(4x+9y) - (4x-14y) = -1 - 26$$

$$23y = -27 \implies y = -\frac{27}{23}$$

Substitute $y = -\frac{27}{23}$ into $(**)$:

$$2x = 13 + 7y = 13 + 7\left(-\frac{27}{23}\right)$$

$$2x = \frac{13 \cdot 23}{23} - \frac{189}{23} = \frac{299-189}{23} = \frac{110}{23}$$

$$x = \frac{110}{2 \cdot 23} = \frac{55}{23}$$

Answer: $x = \frac{55}{23}, y = -\frac{27}{23}$

12. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcenter of the triangle. Then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

Solution: Let z_0 be the circumcenter. For an equilateral triangle, the circumcenter is also the centroid.

$$z_0 = \frac{z_1+z_2+z_3}{3} \implies z_1 + z_2 + z_3 = 3z_0 \quad (*)$$

The condition for z_1, z_2, z_3 to form an equilateral triangle is $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ (since the origin is not the center).

The vertices z_1, z_2, z_3 form an equilateral triangle with center z_0 if and only if $z_1 - z_0, z_2 - z_0, z_3 - z_0$ are the vertices of an equilateral triangle centered at the origin, which means they are related by the cube roots of unity.

$$(z_1 - z_0) + (z_2 - z_0)\omega + (z_3 - z_0)\omega^2 = 0 \quad \text{or}$$

$$(z_1 - z_0)^2 + (z_2 - z_0)^2 + (z_3 - z_0)^2 = (z_1 - z_0)(z_2 - z_0) + (z_2 - z_0)(z_3 - z_0) + (z_3 - z_0)(z_1 - z_0)$$

Using the center at the origin condition for $Z_1 = z_1 - z_0, Z_2 = z_2 - z_0, Z_3 = z_3 - z_0$:

$$Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$$

Also, since the center is the centroid:

$$Z_1 + Z_2 + Z_3 = (z_1 + z_2 + z_3) - 3z_0 = 3z_0 - 3z_0 = 0$$

The identity $(Z_1 + Z_2 + Z_3)^2 = Z_1^2 + Z_2^2 + Z_3^2 + 2(Z_1Z_2 + Z_2Z_3 + Z_3Z_1)$ implies:

$$0^2 = 3(Z_1^2 + Z_2^2 + Z_3^2)$$

This identity implies $Z_1^2 + Z_2^2 + Z_3^2 = 0$ only if $Z_1 = Z_2 = Z_3 = 0$. This is incorrect.
Let's use the condition $Z_1 + Z_2 + Z_3 = 0$. Squaring this:

$$(Z_1 + Z_2 + Z_3)^2 = 0$$

$$Z_1^2 + Z_2^2 + Z_3^2 + 2(Z_1Z_2 + Z_2Z_3 + Z_3Z_1) = 0$$

Since Z_1, Z_2, Z_3 form an equilateral triangle:

$$Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$$

Substitute the second equation into the first:

$$Z_1^2 + Z_2^2 + Z_3^2 + 2(Z_1^2 + Z_2^2 + Z_3^2) = 0$$

$$3(Z_1^2 + Z_2^2 + Z_3^2) = 0 \implies Z_1^2 + Z_2^2 + Z_3^2 = 0$$

Now expand $Z_k = z_k - z_0$:

$$(z_1 - z_0)^2 + (z_2 - z_0)^2 + (z_3 - z_0)^2 = 0$$

$$(z_1^2 - 2z_1z_0 + z_0^2) + (z_2^2 - 2z_2z_0 + z_0^2) + (z_3^2 - 2z_3z_0 + z_0^2) = 0$$
$$(z_1^2 + z_2^2 + z_3^2) - 2z_0(z_1 + z_2 + z_3) + 3z_0^2 = 0$$

Substitute $z_1 + z_2 + z_3 = 3z_0$ from $(*)$:

$$(z_1^2 + z_2^2 + z_3^2) - 2z_0(3z_0) + 3z_0^2 = 0$$

$$(z_1^2 + z_2^2 + z_3^2) - 6z_0^2 + 3z_0^2 = 0$$
$$z_1^2 + z_2^2 + z_3^2 = 3z_0^2$$

13. A relation R on the set of complex numbers is defined by $z_1 R z_2$, if and only if $\frac{z_1 - z_2}{z_1 + z_2}$ is real. Show that R is an equivalence relation.

Solution: A relation R is an equivalence relation if it is reflexive, symmetric, and transitive. The domain of R excludes $z_1 = -z_2$.
The condition for $w = \frac{z_1 - z_2}{z_1 + z_2}$ to be real is $w = \overline{w}$.

$$\frac{z_1 - z_2}{z_1 + z_2} = \overline{\left(\frac{z_1 - z_2}{z_1 + z_2}\right)} = \frac{\overline{z_1} - \overline{z_2}}{\overline{z_1} + \overline{z_2}}$$

Cross-multiplying:

$$(z_1 - z_2)(\overline{z_1} + \overline{z_2}) = (z_1 + z_2)(\overline{z_1} - \overline{z_2})$$

$$z_1\overline{z_1} + z_1\overline{z_2} - z_2\overline{z_1} - z_2\overline{z_2} = z_1\overline{z_1} - z_1\overline{z_2} + z_2\overline{z_1} - z_2\overline{z_2}$$

$$|z_1|^2 + z_1\overline{z_2} - z_2\overline{z_1} - |z_2|^2 = |z_1|^2 - z_1\overline{z_2} + z_2\overline{z_1} - |z_2|^2$$

$$z_1\overline{z_2} - z_2\overline{z_1} = -z_1\overline{z_2} + z_2\overline{z_1}$$

$$2z_1\overline{z_2} = 2z_2\overline{z_1}$$

$$z_1\overline{z_2} = z_2\overline{z_1}$$

This is the simplified condition for $z_1 R z_2$.

1. **Reflexivity (zRz):** We need to show zRz, i.e., $z\overline{z} = z\overline{z}$. This is always true for any z. (Also, $\frac{z-z}{z+z} = \frac{0}{2z} = 0$, which is real, provided $z \neq 0$). If $z = 0$, the relation is $\frac{0}{0}$, which is undefined. Assuming $z_1, z_2 \neq 0$, R is reflexive.

2. **Symmetry ($z_1 R z_2 \implies z_2 R z_1$):** Assume $z_1 R z_2$, so $z_1\overline{z_2} = z_2\overline{z_1}$. We need to show $z_2 R z_1$, so $z_2\overline{z_1} = z_1\overline{z_2}$. This is immediate from the commutative property of multiplication.

3. **Transitivity ($z_1 R z_2$ and $z_2 R z_3 \implies z_1 R z_3$):** Given $z_1 R z_2 \implies z_1\overline{z_2} = z_2\overline{z_1} \quad (*)$ Given $z_2 R z_3 \implies z_2\overline{z_3} = z_3\overline{z_2} \quad (**)$

From $(*)$, if $z_1, z_2 \neq 0$: $\frac{z_1}{z_2} = \frac{\overline{z_1}}{\overline{z_2}} = \overline{(\frac{z_1}{z_2})}$. This means $\frac{z_1}{z_2}$ is real, say λ_1.

$$z_1 = \lambda_1 z_2, \quad \lambda_1 \in \mathbb{R}$$

From $(**)$, if $z_2, z_3 \neq 0$: $\frac{z_2}{z_3}$ is real, say λ_2.

$$z_2 = \lambda_2 z_3, \quad \lambda_2 \in \mathbb{R}$$

Substitute z_2 into the expression for z_1:

$$z_1 = \lambda_1(\lambda_2 z_3) = (\lambda_1\lambda_2)z_3$$

Since $\lambda_1\lambda_2$ is real, $\frac{z_1}{z_3}$ is real, so $z_1\overline{z_3} = z_3\overline{z_1}$. Thus $z_1 R z_3$.

R is reflexive, symmetric, and transitive, so R is an equivalence relation (under the assumption $z_1, z_2 \neq 0$).

14. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1 z_2 = 0$

Solution: Let the vertices be $O(0), A(z_1), B(z_2)$. For the triangle OAB to be equilateral, the side lengths must be equal, and the angle at O must be $60° \ (\pm\frac{\pi}{3})$.

1. **Side lengths equal:**

$$|z_1| = |z_2| = |z_1 - z_2|$$
$$|z_1| = |z_2|$$
$$|z_1|^2 = |z_1 - z_2|^2$$
$$z_1\overline{z_1} = (z_1 - z_2)(\overline{z_1 - z_2}) = (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$
$$|z_1|^2 = z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2}$$
$$|z_1|^2 = |z_1|^2 - z_1\overline{z_2} - z_2\overline{z_1} + |z_2|^2$$

Since $|z_1| = |z_2|$, the equation simplifies to:

$$0 = -z_1\overline{z_2} - z_2\overline{z_1} + |z_1|^2$$

$$z_1\overline{z_2} + z_2\overline{z_1} = |z_1|^2 \quad (*)$$

2. **Angle condition ($\angle AOB = \pm\frac{\pi}{3}$):** The rotation of z_1 by $\pm\frac{\pi}{3}$ about the origin must result in z_2.

$$z_2 = z_1 e^{\pm i\frac{\pi}{3}}$$

The two possible values for $e^{\pm i\frac{\pi}{3}}$ are $\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

$$z_2 = z_1\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \quad \text{or} \quad z_2 = z_1\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

In either case, $\frac{z_2}{z_1} = \frac{1 \pm i\sqrt{3}}{2}$.

This means $\frac{z_2}{z_1}$ is a root of the equation $x^2 - 2\text{Re}(x)x + |x|^2 = 0$. Here $x = \frac{z_2}{z_1}$, $\text{Re}(x) = \frac{1}{2}$, $|x|^2 = 1$.

$$x^2 - 2(\frac{1}{2})x + 1 = 0 \implies x^2 - x + 1 = 0$$

Substitute $x = \frac{z_2}{z_1}$:

$$\left(\frac{z_2}{z_1}\right)^2 - \frac{z_2}{z_1} + 1 = 0$$

Multiply by z_1^2 (since $z_1 \neq 0$):

$$z_2^2 - z_1 z_2 + z_1^2 = 0$$

Thus, the triangle is equilateral if and only if $z_1^2 + z_2^2 - z_1 z_2 = 0$. (Note that the angle condition is sufficient and implies the side length condition, e.g., $|z_2| = |z_1||e^{\pm i\pi/3}| = |z_1| \cdot 1 = |z_1|$).

15. If $1, z_1, z_2, \ldots, z_{n-1}$ are the n roots of unity, then show that $(1-z_1)(1-z_2)(1-z_3)\ldots(1-z_{n-1}) = n$

Solution: The n^{th} roots of unity are the roots of the equation $z^n - 1 = 0$.

We can factor $z^n - 1$ in terms of its roots:

$$z^n - 1 = (z-1)(z-z_1)(z-z_2)\ldots(z-z_{n-1})$$

Since $z \neq 1$, we can divide both sides by $(z-1)$:

$$\frac{z^n - 1}{z-1} = (z-z_1)(z-z_2)\ldots(z-z_{n-1})$$

We use the identity for a geometric series sum: $\frac{z^n-1}{z-1} = 1 + z + z^2 + \ldots + z^{n-1}$.

$$1 + z + z^2 + \ldots + z^{n-1} = (z-z_1)(z-z_2)\ldots(z-z_{n-1})$$

We want to find the value of the expression at $z = 1$. Substitute $z = 1$ into the equation:

$$\text{LHS} = 1 + 1^2 + 1^3 + \ldots + 1^{n-1} = \underbrace{1 + 1 + \ldots + 1}_{n \text{ terms}} = n$$

$$\text{RHS} = (1-z_1)(1-z_2)\ldots(1-z_{n-1})$$

Equating LHS and RHS:

$$(1-z_1)(1-z_2)(1-z_3)\ldots(1-z_{n-1}) = n$$