SET 2 - DETAILED SOLUTIONS

1. A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five questions. The number of choices available to him is

   Solution: Total questions $Q_{Total} = 13$. Questions to be answered $Q_{Ans} = 10$. The questions are divided into two groups: $G_1 = \{Q_1, \dots, Q_5\}$ (5 questions) and $G_2 = \{Q_6, \dots, Q_{13}\}$ (8 questions). The student must choose at least 4 from G_1.

   Let k be the number of questions chosen from G_1. Since 10 questions are chosen in total, $10 - k$ questions must be chosen from G_2. Possible cases for k:

   (i) **Choose 4 from G_1 and 6 from G_2 ($k = 4$):**

   $$C_4^5 \times C_6^8 = 5 \times C_2^8 = 5 \times \frac{8 \times 7}{2} = 5 \times 28 = 140$$

   (ii) **Choose 5 from G_1 and 5 from G_2 ($k = 5$):**

   $$C_5^5 \times C_5^8 = 1 \times C_3^8 = 1 \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 1 \times 56 = 56$$

   Total number of choices $= 140 + 56 = 196$.

   Answer: (c) 196

2. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by

   Solution: This is a problem of circular permutation with restriction. The restriction is that no two women (W) can sit together.

   Step 1: Arrange the 6 men (M). First, seat the 6 men around the circular table. The number of ways to arrange n distinct items in a circle is $(n-1)!$.

   $$N_M = (6-1)! = 5! = 120$$

   Step 2: Place the 5 women. Arranging the 6 men creates 6 spaces between them where the women can be seated.

   $$\_M\_M\_M\_M\_M\_M\_$$

   Since the 5 women must be placed in 5 of these 6 spaces, and the women are distinct, this is a permutation problem.

   $$N_W = P_5^6 = \frac{6!}{(6-5)!} = 6! = 720$$

   Total number of arrangements $N = N_M \times N_W = 5! \times 6!$.

   Checking the options for the required form:

   $$N = 5! \times 6! = (5 \times 4 \times 3 \times 2 \times 1) \times (6 \times 5 \times 4 \times 3 \times 2 \times 1)$$

The question options seem to be in a very simplified form, possibly referring to the general formula structure or a typo. The standard form for this arrangement is $(m-1)! \times P^m_w = (6-1)! \times P^6_5 = 5! \times 6!$.

If we look at the options as expressions, the closest form to the structure 5×6 is **(a)** $7! \times 5!$ which is wrong. The intended structure might be related to the number of seats available, $7 \times 5 = 35$, which is certainly not the right answer. Assuming a major typo in the options and selecting the one closest to the correct calculation logic, but $5! \times 6!$ is the correct answer.

Since the correct answer is $5! \times 6! = 120 \times 720 = 86400$, and none of the options are correct, we must look for a common error structure.

If the women were identical: $5! \times C^6_5 = 120 \times 6 = 720$. If the men were identical: $C^6_5 = 6$ ways.

There is a very high probability of a typo in the options or the intended answer key. However, based on the structure $5! \times 6!$ and no correct option, and the provided answer (a) being $7! \times 5!$, we state the correct mathematical answer. If forced to choose from the options, they are all incorrect for the problem statement.

Answer: $5! \times 6! = 86400$ **(The given options are incorrect for the problem).**

3. How many ways are there to arrange the letters in the word GARDEN with vowels in alphabetical order

   Solution: The word GARDEN has 6 distinct letters. Vowels: A, E (2 vowels). Consonants: G, R, D, N (4 consonants).

   Total number of arrangements of 6 distinct letters is $6! = 720$.

   The vowels must be in alphabetical order (A before E). In any arrangement of the 6 letters, the two vowels (A and E) appear in one of two ways: (A, E) or (E, A). Since only one order (A, E) is allowed, the number of allowed arrangements is half of the total arrangements.

$$N = \frac{\text{Total arrangements}}{\text{Arrangements of Vowels}} = \frac{6!}{2!} = \frac{720}{2} = 360$$

   Alternatively, consider the 6 places. Choose 2 places for the vowels A and E in C^6_2 ways. Once the places are chosen, the vowels must be placed in the order A, E (1 way). The remaining 4 consonants (G, R, D, N) can be arranged in the remaining 4 places in 4! ways.

$$N = C^6_2 \times 1 \times 4! = 15 \times 1 \times 24 = 360$$

   Answer: (c) 360

4. The number of ways of distributing 8 identical balls in 3 distinct boxes so that none of the boxes is empty is

   Solution: This is a problem of distributing identical items into distinct containers with the constraint that no container is empty. This is equivalent to finding the number of positive integer solutions to the equation:

$$x_1 + x_2 + x_3 = 8$$

where $x_1, x_2, x_3 \geq 1$ are the number of balls in the 3 boxes.

We use the Stars and Bars method for positive integer solutions: C^{n-1}_{r-1}, where n is the number of identical balls (8) and r is the number of distinct boxes (3).

$$N = C^{8-1}_{3-1} = C^7_2 = \frac{7 \times 6}{2 \times 1} = 21$$

Answer: (b) 21

5. If the letters of the word SACHIN are arranged in all possible ways and these words are written out as in the dictionary , then the word SACHIN appears at serial number

   Solution: The word SACHIN has 6 distinct letters: A, C, H, I, N, S. The letters in alphabetical order are: **A, C, H, I, N, S**.

   We count the number of words starting with letters alphabetically before S.

   (a) **Words starting with A**: Fix A at the first place. The remaining 5 letters (C, H, I, N, S) can be arranged in 5! ways.

   $$N_A = 5! = 120$$

   (b) **Words starting with C**: $N_C = 5! = 120$
   (c) **Words starting with H**: $N_H = 5! = 120$
   (d) **Words starting with I**: $N_I = 5! = 120$
   (e) **Words starting with N**: $N_N = 5! = 120$

   Total words before those starting with S: $120 \times 5 = 600$.

   The first word starting with S is the one with the remaining letters in alphabetical order: S A C H I N.

   Therefore, the word SACHIN is the **601$^{\text{st}}$** word.

   Answer: (a) 601

6. At an election, a voter may vote for any number of candidates, not greater than the number to be elected. There are 10 candidates and 4 are to be selected, if a voter votes for at least one candidate, then the number of ways in which he can vote is

   Solution: Total candidates $n = 10$. Maximum number to be selected $k = 4$. The voter can vote for r candidates, where $1 \leq r \leq 4$.

   Number of ways to vote for exactly r candidates is C^{10}_r. The condition "votes for at least one candidate" means $r = 1, 2, 3,$ or 4.

   Total number of ways $N = C^{10}_1 + C^{10}_2 + C^{10}_3 + C^{10}_4$.

$$C^{10}_1 = 10$$
$$C^{10}_2 = \frac{10 \times 9}{2} = 45$$
$$C^{10}_3 = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 10 \times 12 = 120$$

$$C_4^{10} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 10 \times 3 \times 7 = 210$$

$$N = 10 + 45 + 120 + 210 = 385$$

Answer: (c) 385

7. The set $S = \{1, 2, 3, ...12\}$ is to be partitioned three sets A,B,C of equal sizes. Thus $A \cup B \cup C = S$, $A \cap B = B \cap C = A \cap C = \phi$. The number of ways to partition S is

   Solution: The set S has $n = 12$ elements. It is partitioned into $k = 3$ sets (A, B, C) of equal size. Since $12/3 = 4$, each set must have $r = 4$ elements.

   The process is:

   (a) Choose 4 elements for set A: C_4^{12} ways.

   (b) Choose 4 elements for set B from the remaining $12 - 4 = 8$ elements: C_4^8 ways.

   (c) Choose 4 elements for set C from the remaining $8 - 4 = 4$ elements: C_4^4 ways.

   Since the sets A, B, and C are **distinct** (labelled), the order of selection matters.

$$N = C_4^{12} \times C_4^8 \times C_4^4 = \frac{12!}{4!(12-4)!} \times \frac{8!}{4!(8-4)!} \times \frac{4!}{4!0!}$$
$$N = \frac{12!}{4!8!} \times \frac{8!}{4!4!} \times \frac{4!}{4!0!} = \frac{12!}{4!4!4!} = \frac{12!}{(4!)^3}$$

   (If the sets were required to be unlabelled, we would divide by 3! since there are 3! ways to arrange the 3 identical groups).

   Answer: (a) $\frac{12!}{(4!)^3}$

8. In a shop there are five types of ice creams available. A child buys six ice creams.

   STATEMENT - I : The number of different ways the child can buy the six ice creams is ${}^{10}C_5$

   STATEMENT - II : The number of different ways the child can buy the six ice creams is equal to the number of different ways of arranging 6 A's and 4 B's in a row.

   Solution: This is a problem of combinations with repetition (identical items: 6 purchases; distinct containers: 5 types). Let $n = 5$ be the number of types (distinct) and $k = 6$ be the number of ice creams bought (identical). The formula for combinations with repetition is C_k^{n+k-1}.

$$N = C_6^{5+6-1} = C_6^{10}$$

   Since $C_6^{10} = C_{10-6}^{10} = C_4^{10}$.

   Evaluation of Statements:

   STATEMENT - I: $N = C_5^{10}$. This is **FALSE**. The correct formula gives C_6^{10} or C_4^{10}.

STATEMENT - II: The number of ways of arranging 6 A's and 4 B's in a row is the number of permutations of $n = 10$ items with $r_1 = 6$ repetitions (A's) and $r_2 = 4$ repetitions (B's).

$$N' = \frac{10!}{6!4!} = C_6^{10} \quad (\text{or } C_4^{10})$$

Since the correct number of ways to buy the ice cream is C_6^{10}, Statement II is the correct calculation of the number of ways. This statement is **TRUE**.

Conclusion: Statement I is False, and Statement II is True.

Answer: (d) Statement - I is False, Statement II is TRUE. (Note: Based on the provided answer key (a), Statement I is considered True. This implies C_5^{10} was intended to be correct, which is mathematically false for C_k^{n+k-1}. Sticking to the correct math, (d) is the right choice).

9. How many different words can be formed by jumbling the letters in the word MISSISSIPPI in which no two S are adjacent ?

   Solution: The word MISSISSIPPI has 11 letters: M(1), I(4), S(4), P(2).

   Step 1: Arrange the letters that are NOT S (The others). The "others" (O) are M(1), I(4), P(2). Total $1 + 4 + 2 = 7$ letters. Number of arrangements $N_O = \frac{7!}{1!4!2!} = \frac{5040}{24\times2} = 105$.

   Step 2: Place the 4 S's in the spaces. Arranging the 7 "other" letters creates $7 + 1 = 8$ possible spaces (including the ends) for the 4 S's.

   $$\_O\_O\_O\_O\_O\_O\_O\_$$

   We must choose 4 of these 8 spaces to place the 4 identical S's. Number of ways to place the S's $N_S = C_4^8$.

   Total number of words $N = N_O \times N_S$.

   $$N = \frac{7!}{4!2!} \times C_4^8 = 105 \times 70 = 7350$$

   Now we match this with the given options, where $C_4^8 = 70$ and $\frac{7!}{4!2!} = 105$.

   - (a) $8 \cdot C_4^6 \cdot C_4^7 = 8 \cdot 15 \cdot 35 = 4200$
   - (d) $7 \cdot C_4^6 \cdot C_4^8 = 7 \cdot 15 \cdot 70 = 7350$

   The correct result 7350 matches the form of option (d), $7 \cdot C_4^6 \cdot C_4^8$, although the coefficient of 7 is strange, and $C_4^6 = 15$ is not the number of arrangements of the other letters. The intended structure is likely $\frac{7!}{4!2!} \times C_4^8 = 105 \times C_4^8$. The option (d) is the only one that evaluates to 7350.

   Answer: (d) $7 \cdot C_4^6 \cdot C_4^8$

10. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then the number of such arrangement is

    Solution: The arrangement is a row of 5 books (4 novels + 1 dictionary). The dictionary must be in the middle (3rd position).

    Step 1: Select the books.

(a) Select 4 novels from 6: C_4^6 ways.

$$C_4^6 = \frac{6 \times 5}{2} = 15$$

(b) Select 1 dictionary from 3: C_1^3 ways.

$$C_1^3 = 3$$

Number of ways to select the 5 books: $15 \times 3 = 45$.

Step 2: Arrange the books. The 5 books are arranged in 5 positions, with the dictionary (D) fixed in the middle position (3).

$$\_\_\mathbf{D}\_\_$$

The 4 selected novels must be arranged in the remaining 4 positions. Since the novels are different, this is a permutation of 4 distinct books.

$$4! = 24$$

Step 3: Total arrangements.

$$N = \text{(Selection ways)} \times \text{(Arrangement ways)} = 45 \times 24 = 1080$$

Checking the options:

(a) at least 500 but less than 750 (No)

(b) at least 750 but less than 1000 (No)

(c) at least 1000 (Yes)

(d) less than 500 (No)

Answer: (c) at least 1000

11. In a certain test, a_i students gave wrong answers to at least i questions where $i = 1, 2, 3, ...k$. No student gave more than k wrong answers. The total number of wrong answer given is.......

Solution: This is a standard application of the "Layer Counting" Principle in combinatorics.

- a_1 students gave ≥ 1 wrong answer.
- a_2 students gave ≥ 2 wrong answers.
- ...
- a_k students gave $\geq k$ wrong answers.
- a_{k+1} (which is 0) students gave $\geq k + 1$ wrong answers.

Consider a student who gave exactly j wrong answers ($1 \leq j \leq k$). This student is counted in the groups $a_1, a_2, \dots, a_j$, but not in $a_{j+1}, \dots, a_k$. Thus, a student with j wrong answers contributes to j of the counts a_i.

The total number of wrong answers given is the sum of the number of wrong answers by each student. If we sum the a_i's, each student who made j wrong answers will be counted exactly j times.

Total number of wrong answers $= \sum_{i=1}^{k} a_i$.

Answer: $\sum_{i=1}^{k} a_i$

12. Let A be a set of n distinct elements. Then the total number of distinct functions from A to A is .... and out of these... are onto function

    Solution: Let $A = \{a_1, a_2, \dots, a_n\}$. A function $f : A \to A$ is defined by assigning an image in the codomain A to each element in the domain A.

    1. Total number of distinct functions ($f : A \to A$): For each of the n elements in the domain A, there are n choices for its image in the codomain A.

$$N_{\text{Total Functions}} = \underbrace{n \times n \times \cdots \times n}_{n \text{ times}} = n^n$$

    2. Number of onto functions ($f : A \to A$): A function $f : A \to A$ is **onto** (surjective) if the range of the function is equal to the codomain A. When the domain and codomain have the same finite number of elements (n), a function is onto if and only if it is also one-to-one (injective), which means it is a **bijection** (a permutation of the elements).

    The number of bijections from a set of n elements to itself is the number of permutations of the n elements.

$$N_{\text{Onto Functions}} = n!$$

    Answer: n^n, $n!$

13. Total number of ways in which six $'+'$ and four $'-'$ signs can be arranged in a line such that no two $'-'$ signs occur together is......

    Solution: We have 6 identical positive signs $(+)$ and 4 identical negative signs $(-)$. The restriction is that no two negative signs can be adjacent.

    Step 1: Arrange the 6 positive signs. Since the signs are identical, there is only 1 way to arrange them.

$$+ + + + ++$$

    Step 2: Place the 4 negative signs in the spaces. Arranging the 6 positive signs creates $6 + 1 = 7$ spaces where the negative signs can be placed.

$$\_ + \_ + \_ + \_ + \_ + \_ + \_$$

    We need to choose 4 of these 7 spaces to place the 4 negative signs. Since the negative signs are identical, this is a combination problem.

$$N = C_4^7$$

$$C_4^7 = C_3^7 = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

Answer: 35

14. There are four balls of different colours and four boxes of colours, same as those of the balls. The number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour is.........

Solution: This is a problem of **derangements** (permutations where no element remains in its original position). Let $n = 4$ be the number of balls/boxes. Let D_n be the number of derangements of n elements. The formula for D_n is:

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

For $n = 4$:

$$D_4 = 4! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$D_4 = 24 \left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)$$

$$D_4 = 24 \left(\frac{12}{24} - \frac{4}{24} + \frac{1}{24}\right) = 24 \left(\frac{12 - 4 + 1}{24}\right) = 24 \times \frac{9}{24} = 9$$

Answer: 9

15. The product of any r consecutive natural numbers is always divisible by $r!$ (True/False)

Solution: Let the r consecutive natural numbers be $n, n+1, \ldots, n+r-1$. Their product is $P = n(n+1)\ldots(n+r-1)$.

Consider the binomial coefficient C_r^{n+r-1}:

$$C_r^{n+r-1} = \frac{(n+r-1)!}{r!(n-1)!} = \frac{(n+r-1)(n+r-2)\ldots(n)}{r!}$$

The numerator is exactly the product P:

$$P = (n+r-1)(n+r-2)\ldots(n)$$

Since C_r^{n+r-1} is an integer (it represents the number of ways to choose r items from $n+r-1$ items), it means that $\frac{P}{r!}$ is an integer.

Therefore, the product of any r consecutive natural numbers is always divisible by $r!$.

Answer: True

16. Ten different letters of an alphabet are given. Words with five letters are formed from three given letters. Then the number of words which have at least one letter repeated as :

Solution: This problem is slightly confusingly worded ("Words with five letters are formed from three given letters"). Assuming the intended meaning is:

$n = 10$ different letters are available. Words of length $r = 5$ are formed using these 10 letters (repetition allowed). We need to find the number of words with **at least one letter repeated**.

1. Total number of 5-letter words (repetition allowed): For each of the 5 positions, there are 10 choices.

$$N_{Total} = 10 \times 10 \times 10 \times 10 \times 10 = 10^5 = 100000$$

2. Number of words with NO letter repeated (all distinct): This is the number of permutations of 10 items taken 5 at a time.

$$N_{Distinct} = P_5^{10} = 10 \times 9 \times 8 \times 7 \times 6 = 30240$$

3. Number of words with at least one letter repeated:

$$N_{Repeated} = N_{Total} - N_{Distinct}$$

$$N_{Repeated} = 100000 - 30240 = 69760$$

Answer: (a) 69760

17. The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to

Solution: Let $E = C_4^{47} + \sum_{j=1}^{5} C_3^{52-j}$.

Expand the sum:

$$\sum_{j=1}^{5} C_3^{52-j} = C_3^{51} + C_3^{50} + C_3^{49} + C_3^{48} + C_3^{47}$$

The expression E is:

$$E = C_4^{47} + C_3^{47} + C_3^{48} + C_3^{49} + C_3^{50} + C_3^{51}$$

We repeatedly apply Pascal's identity: $C_r^n + C_{r-1}^n = C_r^{n+1}$.

$$\begin{aligned}
E &= (C_4^{47} + C_3^{47}) + C_3^{48} + C_3^{49} + C_3^{50} + C_3^{51} \\
&= C_4^{48} + C_3^{48} + C_3^{49} + C_3^{50} + C_3^{51} \\
&= (C_4^{48} + C_3^{48}) + C_3^{49} + C_3^{50} + C_3^{51} \\
&= C_4^{49} + C_3^{49} + C_3^{50} + C_3^{51} \\
&= C_4^{50} + C_3^{50} + C_3^{51} \\
&= C_4^{51} + C_3^{51} \\
&= C_4^{52}
\end{aligned}$$

Answer: (c) ${}^{52}C_4$

18. An n- digit number is a positive number with exactly n digits. Nine hundred distinct n digit numbers are to be formed using only the three digits 2,5 and 7. The smallest value of n for which this is possible is

    Solution: The number of distinct n-digit numbers that can be formed using only the digits 2, 5, and 7 is 3^n, since for each of the n positions, there are 3 choices (2, 5, or 7).

    We are given that 900 distinct n-digit numbers are to be formed. This means the total number of possible n-digit numbers must be greater than or equal to 900.

    $$3^n \geq 900$$

    We check powers of 3:

    - $3^1 = 3$
    - $3^2 = 9$
    - $3^3 = 27$
    - $3^4 = 81$
    - $3^5 = 243$
    - $3^6 = 729$ (Too small, $729 < 900$)
    - $3^7 = 2187$ (Sufficient, $2187 \geq 900$)

    The smallest value of n for which $3^n \geq 900$ is $n = 7$.

    Answer: (b) 7

ANSWERS - SET 2 (Provided for reference)

1. c (196)
2. a (Incorrect, $5! \times 6!$)
3. c (360)
4. b (21)
5. a (601)
6. c (385)
7. a ($\frac{12!}{(4!)^3}$)
8. d (Based on correct math)
9. d (7350)
10. c (1080)
11. $\sum_{i=1}^{k} a_i$
12. n^n, $n!$

13. 35
14. 9
15. True
16. a (69760)
17. c ($^{52}C_4$)
18. b (7)