SETS, RELATIONS & FUNCTIONS - SET 4

1. Consider the following statements : (a) The product of two even or odd function is an even function.
(b) The product of an even function and an odd function is an odd function
(c) Every function can be expressed as the sum of an even and an odd function.
Which of the statements given above is / are correct

   (a) only a
   (b) only b
   (c) only c
   (d) **all of the above**

   Solution: Let $f_e(x)$ be an even function ($f_e(-x) = f_e(x)$) and $f_o(x)$ be an odd function ($f_o(-x) = -f_o(x)$).

   - **(a) True:**
     - Product of two even functions, $P(x) = f_{e1}(x)f_{e2}(x)$:
     $$P(-x) = f_{e1}(-x)f_{e2}(-x) = f_{e1}(x)f_{e2}(x) = P(x) \quad \text{(Even)}$$
     - Product of two odd functions, $P(x) = f_{o1}(x)f_{o2}(x)$:
     $$P(-x) = f_{o1}(-x)f_{o2}(-x) = (-f_{o1}(x))(-f_{o2}(x)) = f_{o1}(x)f_{o2}(x) = P(x) \quad \text{(Even)}$$
   - **(b) True:** Product of an even and an odd function, $P(x) = f_e(x)f_o(x)$:
   $$P(-x) = f_e(-x)f_o(-x) = f_e(x)(-f_o(x)) = -(f_e(x)f_o(x)) = -P(x) \quad \text{(Odd)}$$
   - **(c) True:** Any function $f(x)$ can be uniquely expressed as the sum of an even function $f_e(x)$ and an odd function $f_o(x)$:
   $$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{Even part, } f_e(x)} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{Odd part, } f_o(x)}$$

   All three statements are correct.

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2. If A = $\{x : f(x) = 0\}$ and B = $\{x : g(x) = 0\}$ then $A \cap B$ will be :

   (a) $[f(x)]^2 + [g(x)]^2 = 0$
   (b) $\frac{f(x)}{g(x)}$
   (c) $\frac{g(x)}{f(x)}$
   (d) none of these

   Solution:

   - A is the set of roots of $f(x) = 0$.
   - B is the set of roots of $g(x) = 0$.
   - $A \cap B$ is the set of values of x for which **both** $f(x) = 0$ and $g(x) = 0$.

Consider the equation $[f(x)]^2 + [g(x)]^2 = 0$. Since $[f(x)]^2 \geq 0$ and $[g(x)]^2 \geq 0$, their sum can only be zero if and only if both terms are zero simultaneously:

$$[f(x)]^2 = 0 \quad \text{and} \quad [g(x)]^2 = 0$$

$$\iff f(x) = 0 \quad \text{and} \quad g(x) = 0$$

Thus, the solution set of $[f(x)]^2+[g(x)]^2 = 0$ is exactly the set $A \cap B$. The other options are expressions, not equations defining a set of roots.

3. In a certain town 25% families own a phone and 15% own a car, 65% families own neither a phone nor a car, 2000 families own both a car and a phone. Consider the following statements in this regard :
(a) 10% families own both a car and a phone
(b) 35% families own either a car or a phone
(c) 40000 families live in town
Which of the above statements are correct

   (a) a and b
   (b) a and c
   (c) **b and c**
   (d) all the above

Solution: Let P be the set of families owning a phone and C be the set of families owning a car. Let N be the total number of families. Given percentages (based on N):

$$P(\text{Phone}) = P(P) = 25\% = 0.25$$

$$P(\text{Car}) = P(C) = 15\% = 0.15$$

$$P(\text{Neither}) = P(P' \cap C') = 65\% = 0.65$$

Statement (b): Families owning either a car or a phone. Using De Morgan's Law: $P(P' \cap C') = P((P \cup C)') = 1 - P(P \cup C)$.

$$P(P \cup C) = 1 - P(P' \cap C') = 1 - 0.65 = 0.35$$

So, 35% families own either a car or a phone. **Statement (b) is correct.**

Statement (a): Families owning both a car and a phone. Using the formula $P(P \cup C) = P(P) + P(C) - P(P \cap C)$:

$$0.35 = 0.25 + 0.15 - P(P \cap C)$$

$$0.35 = 0.40 - P(P \cap C)$$

$$P(P \cap C) = 0.40 - 0.35 = 0.05$$

So, 5% families own both a car and a phone. **Statement (a) is incorrect.**

Statement (c): Total number of families. We are given that 2000 families own both, and we found this corresponds to 5%:

$$0.05 \times N = 2000$$

$$N = \frac{2000}{0.05} = \frac{2000}{\frac{5}{100}} = 2000 \times \frac{100}{5} = 400 \times 100 = 40000$$

So, 40000 families live in town. **Statement (c) is correct.**

The correct statements are (b) and (c).

4. Let N denotes the set of all natural numbers and R be the relation on $N \times N$ defined by (a,b)R(c,d) if ad(b+c)=bc(a+d), then R is ;

   (a) symmetric only

   (b) reflexive only

   (c) transitive

   (d) **an equivalence relation**

   Solution: The relation is given by $ad(b+c) = bc(a+d)$. Let's try to simplify this expression to a more manageable form. Since $a, b, c, d \in \mathbb{N}$, we can divide by $abcd$:

$$\frac{ad(b+c)}{abcd} = \frac{bc(a+d)}{abcd}$$
$$\frac{b+c}{bc} = \frac{a+d}{ad}$$
$$\frac{b}{bc} + \frac{c}{bc} = \frac{a}{ad} + \frac{d}{ad}$$
$$\frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a}$$

   The relation is equivalent to: $(a, b)R(c, d) \iff \frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d}$.
   Let $f(a, b) = \frac{1}{a} + \frac{1}{b}$. The relation is $(a, b)R(c, d) \iff f(a, b) = f(c, d)$.

   - **Reflexive:** $(a, b)R(a, b) \iff f(a, b) = f(a, b)$. This is always true. R is reflexive.
   - **Symmetric:** $(a, b)R(c, d) \implies f(a, b) = f(c, d)$. Then $f(c, d) = f(a, b)$, which means $(c, d)R(a, b)$. R is symmetric.
   - **Transitive:** $(a, b)R(c, d)$ and $(c, d)R(e, f)$.

$$(a, b)R(c, d) \implies f(a, b) = f(c, d)$$
$$(c, d)R(e, f) \implies f(c, d) = f(e, f)$$

     By equality, $f(a, b) = f(e, f)$, which means $(a, b)R(e, f)$. R is transitive.

   Since R is reflexive, symmetric, and transitive, it is an **equivalence relation**.

5. The solution set of $8x \equiv 6(\text{mod} 14), x \in Z$ are

   (a) $[8] \cup [6]$

   (b) $[8] \cup [14]$

   (c) $[6] \cup [13]$

   (d) $[8] \cup [6] \cup [13]$

   Solution: The congruence $ax \equiv b(\text{mod} m)$ has solutions if and only if $\gcd(a, m)$ divides b. Here, $a = 8$, $b = 6$, $m = 14$.

$$d = \gcd(8, 14) = 2$$

   Since $d = 2$ divides $b = 6$, there are $d = 2$ solutions modulo $m = 14$. The congruence $8x \equiv 6(\text{mod} 14)$ is equivalent to the linear Diophantine equation:

$$8x - 6 = 14k \quad \text{for some integer } k$$

Divide the entire equation by $d = 2$:

$$4x - 3 = 7k \implies 4x \equiv 3(\text{mod}7)$$

We need to find the inverse of 4 modulo 7. $4 \times 2 = 8 \equiv 1(\text{mod}7)$. So, $4^{-1} \equiv 2(\text{mod}7)$. Multiply the reduced congruence by 2:

$$2(4x) \equiv 2(3)(\text{mod}7)$$

$$8x \equiv 6(\text{mod}7)$$

$$x \equiv 6(\text{mod}7)$$

The solutions modulo 14 are found by $x = 7j + 6$.

- For $j = 0$: $x_1 = 7(0) + 6 = 6$.
- For $j = 1$: $x_2 = 7(1) + 6 = 13$.

The solutions are $x \equiv 6(\text{mod}14)$ and $x \equiv 13(\text{mod}14)$. The solution set is the union of the congruence classes:

$$[6]_{14} \cup [13]_{14}$$

(Note: The question uses the notation $[x]$ for the congruence class modulo m, where $m = 14$ is implicit).

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6. Let L be the set of all straight lines in the Euclidean plane. Two lines l_1 and l_2 are said to be related by the relation R, iff l_1 is parallel to l_2. Then the relation R is

   (a) reflexive

   (b) symmetric

   (c) transitive

   (d) **equivalence**

   Solution: The relation R is $l_1 R l_2 \iff l_1$ is parallel to l_2 (denoted $l_1 || l_2$).

   - **Reflexive:** A line l_1 is always parallel to itself ($l_1 || l_1$). R is reflexive.
   - **Symmetric:** If l_1 is parallel to l_2 ($l_1 || l_2$), then l_2 is parallel to l_1 ($l_2 || l_1$). R is symmetric.
   - **Transitive:** If l_1 is parallel to l_2 ($l_1 || l_2$) and l_2 is parallel to l_3 ($l_2 || l_3$), then l_1 is parallel to l_3 ($l_1 || l_3$). R is transitive.

   Since R is reflexive, symmetric, and transitive, it is an **equivalence relation**.

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7. Let A $= \{x :\in R, |x| < 1\}$; B $= \{x :\in R, |x| < 1\}$; and $A \cup B = R - D$, then the set D is :

   (a) $\{x : 1 < x \leq 2\}$

   (b) $\{x : 1 \leq x < 2\}$

   (c) $\{x : 1 \leq x \leq 2\}$

   (d) **none of these**

   Solution: The question appears to have a typo or missing information as sets A and B are identical:

$$A = \{x \in \mathbb{R} : |x| < 1\} = (-1, 1)$$

$$B = \{x \in \mathbb{R} : |x| < 1\} = (-1, 1)$$

   The union is:

$$A \cup B = (-1, 1)$$

We are given $A \cup B = \mathbb{R} \setminus D$, where $\mathbb{R}$ is the set of all real numbers.

$$(-1, 1) = \mathbb{R} \setminus D$$

This means D is the set of all real numbers that are **not** in $(-1, 1)$.

$$D = \mathbb{R} \setminus (-1, 1) = (-\infty, -1] \cup [1, \infty)$$

None of the given options matches this set D. (Note: If the second set was $B = \{x \in \mathbb{R} : |x| < 2\} = (-2, 2)$, then $A \cup B = (-2, 2)$, and $D = (-\infty, -2] \cup [2, \infty)$, which still doesn't match the options).

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8. If X $= \{4^n - 3n - 1 : n \in N\}$ and Y $= \{9(n-1) : n \in N\}$, then $X \cup Y$

   (a) X

   (b) **Y**

   (c) N

   (d) none of these

   Solution: Let's list the first few elements of X and Y by substituting $n = 1, 2, 3, \ldots$:

   - **Set** X: $x_n = 4^n - 3n - 1$

   $$n = 1 : x_1 = 4^1 - 3(1) - 1 = 4 - 3 - 1 = 0$$
   $$n = 2 : x_2 = 4^2 - 3(2) - 1 = 16 - 6 - 1 = 9$$
   $$n = 3 : x_3 = 4^3 - 3(3) - 1 = 64 - 9 - 1 = 54$$
   $$n = 4 : x_4 = 4^4 - 3(4) - 1 = 256 - 12 - 1 = 243$$
   $$X = \{0, 9, 54, 243, \ldots\}$$

   - **Set** Y: $y_n = 9(n-1)$

   $$n = 1 : y_1 = 9(1-1) = 0$$
   $$n = 2 : y_2 = 9(2-1) = 9$$
   $$n = 3 : y_3 = 9(3-1) = 18$$
   $$n = 4 : y_4 = 9(4-1) = 27$$
   $$n = 5 : y_5 = 9(5-1) = 36$$
   $$n = 6 : y_6 = 9(6-1) = 45$$
   $$n = 7 : y_7 = 9(7-1) = 54$$
   $$Y = \{0, 9, 18, 27, 36, 45, 54, \ldots\}$$

   It is a known result that $4^n - 3n - 1$ is always divisible by 9. This means every element of X is a multiple of 9. Since Y is the set of all non-negative multiples of 9, X is a subset of Y, i.e., $X \subseteq Y$. (Proof: By Binomial Theorem, $4^n = (1+3)^n = \sum_{k=0}^{n} \binom{n}{k} 3^k = \binom{n}{0} 3^0 + \binom{n}{1} 3^1 + \binom{n}{2} 3^2 + \ldots$ $4^n = 1 + 3n + 9\binom{n}{2} + \ldots$ $4^n - 3n - 1 = 9\left[\binom{n}{2} + \binom{n}{3} 3 + \ldots\right]$. This shows $4^n - 3n - 1$ is a multiple of 9.)

   Since $X \subseteq Y$, the union $X \cup Y$ is equal to the larger set Y.

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9. The number of elements in the set $\{(a, b) : 2a^2 + 3b^2 = 35, a, b \in Z\}$ where Z is the set of all integers, is :

   (a) 2

(b) 4

(c) **8**

(d) 12

Solution: We are looking for integer solutions (a, b) to the equation $2a^2 + 3b^2 = 35$. Since $a^2 \geq 0$ and $b^2 \geq 0$, we can find the bounds for a and b:

- $2a^2 \leq 35 \implies a^2 \leq 17.5$. Since $a \in \mathbb{Z}$, the possible values for $|a|$ are $0, 1, 2, 3, 4$.
- $3b^2 \leq 35 \implies b^2 \leq \frac{35}{3} \approx 11.67$. Since $b \in \mathbb{Z}$, the possible values for $|b|$ are $0, 1, 2, 3$.

We test the possible values for $|a|$ and check if b is an integer:

$$3b^2 = 35 - 2a^2$$

- If $|a| = 0$ $(a = 0)$: $3b^2 = 35 - 0 = 35$. $b^2 = \frac{35}{3}$, not an integer. No solution.
- If $|a| = 1$ $(a = \pm 1)$: $3b^2 = 35 - 2(1)^2 = 33$. $b^2 = 11$, not an integer. No solution.
- If $|a| = 2$ $(a = \pm 2)$: $3b^2 = 35 - 2(2)^2 = 35 - 8 = 27$. $b^2 = 9$. $b = \pm 3$.
  - Solutions: $(2, 3), (2, -3), (-2, 3), (-2, -3)$. (**4 pairs**)
- If $|a| = 3$ $(a = \pm 3)$: $3b^2 = 35 - 2(3)^2 = 35 - 18 = 17$. $b^2 = \frac{17}{3}$, not an integer. No solution.
- If $|a| = 4$ $(a = \pm 4)$: $3b^2 = 35 - 2(4)^2 = 35 - 32 = 3$. $b^2 = 1$. $b = \pm 1$.
  - Solutions: $(4, 1), (4, -1), (-4, 1), (-4, -1)$. (**4 pairs**)

The total number of elements (ordered pairs) in the set is 4+4 = 8.

10. If A = $\{x : x$ is a multiple of $4\}$ and B = $\{x : x$ is a multiple of $6\}$, then $A \cap B$ consists of all multiples of :

(a) 16

(b) **12**

(c) 8

(d) 4

Solution: A is the set of multiples of 4, denoted $4\mathbb{Z}$. B is the set of multiples of 6, denoted $6\mathbb{Z}$. $A \cap B$ is the set of numbers that are multiples of both 4 and 6. This is the set of common multiples of 4 and 6. The set of all common multiples of two numbers is the set of all multiples of their **Least Common Multiple (LCM)**.

$$\text{LCM}(4, 6) = 12$$

Therefore, $A \cap B$ consists of all multiples of 12.

11. Consider the following relation : (a) $A - B = A - (A \cap B)$ (b) $A = (A \cap B) \cup (A - B)$ (c) $A - (B \cup C) = (A - B) \cup (A - C)$ Which of the above statements is/are correct :

(a) **a and b**

(b) b only

(c) b and c

(d) a and b

Solution:

- **(a) True:** The definition of set difference $A - B$ is the set of elements in A that are **not** in B. $A \cap B$ is the part of A that is in B. $A - (A \cap B)$ is the set of elements in A that are **not** in $A \cap B$. This is precisely the part of A that is not in B. $A - B = A \cap B^c$. $A - (A \cap B) = A \cap (A \cap B)^c = A \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c) = \phi \cup (A \cap B^c) = A \cap B^c = A - B$.
- **(b) True:** The set A can be partitioned into two disjoint parts: the part that overlaps with B ($A \cap B$) and the part that does not overlap with B ($A - B$). $(A \cap B) \cap (A - B) = (A \cap B) \cap (A \cap B^c) = A \cap (B \cap B^c) = A \cap \phi = \phi$. Their union must equal A.
- **(c) False (De Morgan's Law for set difference):**

$$A - (B \cup C) = A \cap (B \cup C)^c = A \cap (B^c \cap C^c) = (A \cap B^c) \cap (A \cap C^c) = (A - B) \cap (A - C)$$

The correct statement is $A - (B \cup C) = (A - B) \cap (A - C)$. The statement given is $A - (B \cup C) = (A - B) \cup (A - C)$, which is incorrect. For a counterexample, let $A = \{1, 2\}, B = \{1\}, C = \{2\}$. $A - (B \cup C) = A - \{1, 2\} = \phi$. $(A - B) \cup (A - C) = \{2\} \cup \{1\} = \{1, 2\}$. $\phi \neq \{1, 2\}$.

Statements (a) and (b) are correct.

12. The relation R $= \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on set A $= \{1, 2, 3\}$ is

(a) **reflexive but not symmetric**
(b) reflexive but not transitive
(c) symmetric and transitive
(d) neither symmetric nor transitive

Solution: The set is $A = \{1, 2, 3\}$.

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$$

- **Reflexive:** $(1, 1), (2, 2), (3, 3)$ are all in R. R is reflexive.
- **Symmetric:** If $(a, b) \in R$, must $(b, a) \in R$? $(1, 2) \in R$, but $(2, 1) \notin R$. R is **not symmetric**.
- **Transitive:** If $(a, b) \in R$ and $(b, c) \in R$, must $(a, c) \in R$?
  - $(1, 2) \in R$ and $(2, 3) \in R \implies (1, 3) \in R$. (Yes)
  - The diagonal elements (a, a) and (a, b) or (b, a) or (a, a) and (a, a) are always transitive.

  R is transitive.

The relation is **reflexive and transitive, but not symmetric**. Since only one option is "reflexive but not symmetric", we choose that.

13. The relation "less than" in the set of natural numbers is

(a) only symmetric
(b) **only transitive**
(c) only reflexive
(d) equivalence relation

Solution: The relation R on $\mathbb{N}$ is defined by $xRy \iff x < y$.

- **Reflexive:** $x < x$ is false for all x. R is **not reflexive**.
- **Symmetric:** If $x < y$, then $y < x$ is false. R is **not symmetric**.
- **Transitive:** If $x < y$ and $y < z$, then $x < z$. This is true. R is **transitive**.

The relation is transitive, but not reflexive or symmetric. It is also anti-symmetric since $x < y$ and $y < x$ is never true. It is neither an equivalence relation nor a partial order (since it's not reflexive). Of the given options, only "transitive" is true.

14. Let R be a relation on N defined by x +2y=8. The domain of R is :

    (a) $\{2, 4, 8\}$

    (b) $\{2, 4, 6, 8\}$

    (c) $\{2, 4, 6\}$

    (d) $\{1, 2, 3, 4\}$

    Solution: The relation is $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x + 2y = 8\}$. The domain of R is the set of all possible first components (x) of the ordered pairs in R.

    $$x = 8 - 2y$$

    Since x and y must be natural numbers ($\mathbb{N} = \{1, 2, 3, \dots\}$), we test values for y:

    - $y = 1$: $x = 8 - 2(1) = 6$. $(6, 1) \in R$.
    - $y = 2$: $x = 8 - 2(2) = 4$. $(4, 2) \in R$.
    - $y = 3$: $x = 8 - 2(3) = 2$. $(2, 3) \in R$.
    - $y = 4$: $x = 8 - 2(4) = 0$. $0 \notin \mathbb{N}$. Stop.

    The possible values for x are $6, 4, 2$. The domain of R is $\{2, 4, 6\}$.

15. If R $= \{(x, y) | x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation in Z, then domain of R is :

    (a) $\{0, 1, 2\}$

    (b) $\{0, -1, -2\}$

    (c) $\{-2, -1, 0, 1, 2\}$

    (d) none of these

    Solution: The domain of R is the set of all possible first components (x) of the ordered pairs $(x, y) \in R$. The condition is $x^2 + y^2 \leq 4$, where x, y are integers ($\mathbb{Z}$). To find the domain, we need to find all integers x for which there exists at least one integer y such that the inequality holds. Since $y^2 \geq 0$, the inequality implies $x^2 \leq x^2 + y^2 \leq 4$.

    $$x^2 \leq 4$$

    Since x is an integer, the possible values for x^2 are $0, 1, 4$.

    - If $x^2 = 0$, then $x = 0$.
    - If $x^2 = 1$, then $x = \pm 1$.
    - If $x^2 = 4$, then $x = \pm 2$.

    The domain of R is the set of all such x values: $\{-2, -1, 0, 1, 2\}$. (We can verify that for each of these x, a corresponding $y \in \mathbb{Z}$ exists, e.g., $y = 0$:

    - $x = -2 : (-2)^2 + 0^2 = 4 \leq 4$.
    - $x = -1 : (-1)^2 + 0^2 = 1 \leq 4$.
    - $x = 0 : 0^2 + 0^2 = 0 \leq 4$.

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16. The relation ”is a subset of” on the power set P(A) of set A is :

    (a) symmetric
    (b) **anti symmetric**
    (c) equivalence
    (d) none of these

    Solution: The relation R on $P(A)$ is defined by $XRY \iff X \subseteq Y$, where $X, Y \in P(A)$.

    - **Reflexive:** $X \subseteq X$ is true for all $X \in P(A)$. R is reflexive.
    - **Symmetric:** If $X \subseteq Y$, must $Y \subseteq X$? Let $A = \{1, 2\}$, $X = \{1\}, Y = \{1, 2\}$. $X \subseteq Y$, but $Y \nsubseteq X$. R is **not symmetric**.
    - **Anti-symmetric:** If $X \subseteq Y$ and $Y \subseteq X$, then X must equal Y. This is the definition of set equality. R is **anti-symmetric**.
    - **Transitive:** If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$. This is true. R is transitive.

    The relation is reflexive, anti-symmetric, and transitive, meaning it is a **partial order relation**. Of the given options, the only true property is anti-symmetric.

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17. If X = $\{8^n - 7n - 1 : n \in N\}$ and Y = $\{49(n - 1) : n \in N\}$ then

    (a) $X \subseteq Y$
    (b) $Y \subseteq X$
    (c) $X = Y$
    (d) none of these

    Solution: Let's list the first few elements of X and Y by substituting $n = 1, 2, 3, \ldots$:

    - **Set** X: $x_n = 8^n - 7n - 1$

    $$n = 1 : x_1 = 8^1 - 7(1) - 1 = 8 - 7 - 1 = 0$$
    $$n = 2 : x_2 = 8^2 - 7(2) - 1 = 64 - 14 - 1 = 49$$
    $$n = 3 : x_3 = 8^3 - 7(3) - 1 = 512 - 21 - 1 = 490$$
    $$n = 4 : x_4 = 8^4 - 7(4) - 1 = 4096 - 28 - 1 = 4067$$
    $$X = \{0, 49, 490, 4067, \ldots\}$$

    - **Set** Y: $y_n = 49(n - 1)$

    $$n = 1 : y_1 = 49(1 - 1) = 0$$
    $$n = 2 : y_2 = 49(2 - 1) = 49$$
    $$n = 3 : y_3 = 49(3 - 1) = 98$$
    $$n = 4 : y_4 = 49(4 - 1) = 147$$
    $$n = 5 : y_5 = 49(5 - 1) = 196$$
    $$n = 6 : y_6 = 49(6 - 1) = 245$$
    $$n = 7 : y_7 = 49(7 - 1) = 294$$
    $$n = 8 : y_8 = 49(8 - 1) = 343$$
    $$n = 9 : y_9 = 49(9 - 1) = 392$$
    $$n = 10 : y_{10} = 49(10 - 1) = 441$$
    $$n = 11 : y_{11} = 49(11 - 1) = 490$$
    $$Y = \{0, 49, 98, 147, 196, 245, 294, 343, 392, 441, 490, \ldots\}$$

It is a known result that $8^n - 7n - 1$ is always divisible by 49. (Proof: By Binomial Theorem, $8^n = (1+7)^n = \sum_{k=0}^{n} \binom{n}{k} 7^k = \binom{n}{0}7^0 + \binom{n}{1}7^1 + \binom{n}{2}7^2 + \ldots$

$$8^n = 1 + 7n + 49\binom{n}{2} + 49\sum_{k=3}^{n}\binom{n}{k}7^{k-2}$$

$$8^n - 7n - 1 = 49\binom{n}{2} + 49(\ldots)$$

This shows $8^n - 7n - 1$ is a multiple of 49.)

Since Y is the set of all non-negative multiples of 49, and every element of X is a non-negative multiple of 49, we have $X \subseteq Y$. $X \neq Y$ because, for example, $98 \in Y$ but $98 \notin X$ (since $x_3 = 490$ is the next element after $x_2 = 49$).

SET 4 - TRUE/FALSE AND FUNCTION PROPERTIES

1. If f(x) =(a$-x^n)^{\frac{1}{n}}$ where $a > 0$ and n is a positive integer, then $f[f(x)] = x$

   Solution: TRUE

$$f[f(x)] = f\left((a - x^n)^{\frac{1}{n}}\right)$$

$$= \left(a - \left[(a - x^n)^{\frac{1}{n}}\right]^n\right)^{\frac{1}{n}}$$

$$= (a - (a - x^n))^{\frac{1}{n}}$$

$$= (a - a + x^n)^{\frac{1}{n}}$$

$$= (x^n)^{\frac{1}{n}} = x$$

   (Assuming the domain is restricted such that $f(x)$ is defined and x is real, which is true for the given conditions).

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2. The function $f(x) = \frac{x^2+4x+30}{x^2-8x+18}$ is not one to one.

   Solution: TRUE To check if a function is one-to-one, we check if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. It is often easier to check if it is *not* one-to-one by showing that $f(x) = c$ (a constant) has more than one solution for some value of c. The function is of the form $y = \frac{x^2+4x+30}{x^2-8x+18}$. Rearranging to find x in terms of y:

$$y(x^2 - 8x + 18) = x^2 + 4x + 30$$

$$yx^2 - 8yx + 18y = x^2 + 4x + 30$$

$$(y-1)x^2 - (8y+4)x + (18y - 30) = 0$$

   Since this is a quadratic equation in x (for $y \neq 1$), a single value of y will generally give two values of x, meaning the function is **not one-to-one** (many-to-one). For example, if we choose $y = 3$, the equation is:

$$(3-1)x^2 - (8(3)+4)x + (18(3) - 30) = 0$$

$$2x^2 - 28x + 24 = 0$$

$$x^2 - 14x + 12 = 0$$

   The discriminant is $\Delta = (-14)^2 - 4(1)(12) = 196 - 48 = 148 > 0$, so there are two distinct real solutions for x. Thus, $f(x)$ is **not one-to-one**.

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3. If $f_1(x)$ and $f_2(x)$ are defined on domains D_1 and D_2 respectively, then $f_1(x) + f_2(x)$ is defined on $D_1 \cup D_2$

   Solution: FALSE The sum of two functions, $(f_1+f_2)(x) = f_1(x)+f_2(x)$, is only defined for those values of x where **both** $f_1(x)$ and $f_2(x)$ are defined. Therefore, the domain of f_1+f_2 is $D_1 \cap D_2$. For example, let $f_1(x) = \sqrt{x}$ with $D_1 = [0, \infty)$ and $f_2(x) = \sqrt{1-x}$ with $D_2 = (-\infty, 1]$. $D_1 \cup D_2 = \mathbb{R}$. However, $(f_1+f_2)(x) = \sqrt{x}+\sqrt{1-x}$ is only defined on $D_1 \cap D_2 = [0, 1]$.

4. Which of the following function is periodic:

   (a) $f(x) = x - [x]$ **where [x] denotes the greatest integer less than or equal to the real number x.**

   (b) $f(x) = sin\frac{1}{x}$ for $x \neq 0$, f(0)=0

   (c) f(x) = xcosx

   (d) none of these

   Solution: A function $f(x)$ is **periodic** if there exists a positive number T (the period) such that $f(x+T) = f(x)$ for all x in the domain.

   - **(a) True:** $f(x) = x - [x]$ is the **fractional part function**, often denoted $\{x\}$. We know that $[x+n] = [x] + n$ for any integer n. Let $T = 1$:

     $$f(x+1) = (x+1) - [x+1] = (x+1) - ([x]+1) = x - [x] = f(x)$$

     The function is periodic with a period of 1.
   - **(b) False:** $f(x) = \sin\frac{1}{x}$ is not periodic. Its oscillation becomes infinitely rapid as $x \to 0$.
   - **(c) False:** $f(x) = x\cos x$ is not periodic. Since $f(x+T) = (x+T)\cos(x+T)$, this cannot equal $x \cos x$ for a constant T (the amplitude x is not constant).

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5. For real x, the function $\frac{(x-a)(x-b)}{(x-c)}$ will assume all real values provided :

   (a) $a > b > c$

   (b) $a < b < c$

   (c) $a > c < b$

   (d) $a \leq c \leq b$ **OR** $b \leq c \leq a$

   Solution: Let $y = \frac{(x-a)(x-b)}{(x-c)}$. The function assumes all real values if the range is $(-\infty, \infty)$. Rearranging to a quadratic in x:

   $$y(x-c) = (x-a)(x-b)$$
   $$yx - yc = x^2 - (a+b)x + ab$$
   $$x^2 - (a+b+y)x + (ab+yc) = 0$$

   For y to be in the range, the quadratic equation must have real roots for x. This requires the discriminant D to be non-negative: $D \geq 0$.

   $$D = (a+b+y)^2 - 4(ab+yc) \geq 0$$

   $$(a+b)^2 + 2(a+b)y + y^2 - 4ab - 4yc \geq 0$$
   $$y^2 + [2(a+b) - 4c]y + (a^2 + 2ab + b^2 - 4ab) \geq 0$$
   $$y^2 + 2(a+b-2c)y + (a-b)^2 \geq 0$$

For this quadratic in y to be non-negative for **all real values of** y, the discriminant of this y-quadratic must be less than or equal to zero (and the leading coefficient is $1 > 0$):

$$D_y = [2(a+b-2c)]^2 - 4(1)(a-b)^2 \le 0$$

$$4(a+b-2c)^2 - 4(a-b)^2 \le 0$$

$$(a+b-2c)^2 - (a-b)^2 \le 0$$

Using the difference of squares formula, $X^2 - Y^2 = (X-Y)(X+Y)$:

$$[(a+b-2c)-(a-b)] \cdot [(a+b-2c)+(a-b)] \le 0$$

$$[a+b-2c-a+b] \cdot [a+b-2c+a-b] \le 0$$

$$[2b-2c] \cdot [2a-2c] \le 0$$

$$4(b-c)(a-c) \le 0$$

This inequality holds if and only if $(b-c)$ and $(a-c)$ have opposite signs (or one of them is zero).

- Case 1: $b-c \ge 0$ and $a-c \le 0 \implies c \le b$ and $a \le c$.

$$a \le c \le b$$

- Case 2: $b-c \le 0$ and $a-c \ge 0 \implies b \le c$ and $c \le a$.

$$b \le c \le a$$

The function assumes all real values if $\mathbf{a \le c \le b}$ or $\mathbf{b \le c \le a}$. This is equivalent to saying that c must lie between a and b (inclusive). Option (4) is the correct mathematical condition, although it only lists one half of the condition $a \le c \le b$. Since it's the only option that describes the correct relationship, we choose it.

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6. Let f(x) $= \frac{\alpha x}{x+1}, x \ne -1$. Then for what value of α is $f[f(x)] = x$

   (a) $\sqrt{2}$

   (b) $-\sqrt{2}$

   (c) 1

   (d) **-1**

   Solution: We calculate $f[f(x)]$:

$$f[f(x)] = f\left(\frac{\alpha x}{x+1}\right) = \frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\left(\frac{\alpha x}{x+1}\right)+1}$$

   Multiply the numerator and denominator by $(x+1)$:

$$f[f(x)] = \frac{\alpha^2 x}{\alpha x + (x+1)} = \frac{\alpha^2 x}{(\alpha+1)x+1}$$

   We require $f[f(x)] = x$:

$$\frac{\alpha^2 x}{(\alpha+1)x+1} = x$$

$$\alpha^2 x = x((\alpha+1)x+1)$$

This equation must hold for all x in the domain (i.e., $x \neq 0$ and $x \neq -1$). We can divide by x:

$$\alpha^2 = (\alpha + 1)x + 1$$

$$(\alpha + 1)x = \alpha^2 - 1$$

This equation must hold for all x. This is only possible if the coefficient of x is zero, and the constant term is also zero:

- **Coefficient of x is 0**: $\alpha + 1 = 0 \implies \alpha = -1$.
- **Constant term is 0**: $\alpha^2 - 1 = 0 \implies (\alpha - 1)(\alpha + 1) = 0 \implies \alpha = 1$ or $\alpha = -1$.

The value of α that satisfies both conditions is $\alpha = -\mathbf{1}$.

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7. Suppose f(x) = $(x+1)^2$ for $x \geq -1$. If g(x) is the function whose graph is reflection of the graph of f(x) with respect to the line y =x, then g(x) equals :

   (a) $-\sqrt{x} - 1, x \geq 0$

   (b) $\frac{1}{(x+1)^2}, x > -1$

   (c) $\sqrt{x+1}, x \geq -1$

   (d) $\sqrt{x} - 1, x \geq 0$

Solution: The reflection of the graph of $f(x)$ with respect to the line $y = x$ is the graph of the **inverse function**, $g(x) = f^{-1}(x)$. Let $y = f(x)$, so $y = (x+1)^2$. We need to solve for x in terms of y.

$$y = (x+1)^2$$

$$\sqrt{y} = \pm(x+1)$$

Since the domain of f is $x \geq -1$, we have $x + 1 \geq 0$. This means we must choose the positive square root:

$$\sqrt{y} = x + 1$$

$$x = \sqrt{y} - 1$$

Thus, the inverse function is $g(x) = f^{-1}(x) = \sqrt{x} - 1$.

Now we determine the domain of $g(x)$, which is the range of $f(x)$. Since $x \geq -1$, $x + 1 \geq 0$, so $f(x) = (x+1)^2 \geq 0^2 = 0$. The range of $f(x)$ is $[0, \infty)$, so the domain of $g(x)$ is $\mathbf{x \geq 0}$.

$$g(x) = \sqrt{x} - 1, \quad x \geq 0$$