SOLUTIONS: SETS, RELATIONS & FUNCTIONS - SET 2

Multiple choice questions (detailed solutions)

1. $A = \{x : \; x^2 = 1\} = \{\pm 1\}, \quad B = \{x : \; x^4 = 1\} = \{\pm 1, \pm i\}$.
   The symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ consists of elements in exactly one of the sets. Since $\pm 1 \in A \cap B$, the elements left are $\{i, -i\}$.
   Answer: $\boxed{\{i, -i\}}$.

2. If $A \subseteq B$ then $A \setminus B = \varnothing$ and
   $$A \triangle B = (A \setminus B) \cup (B \setminus A) = B \setminus A.$$
   So $\boxed{B - A}$.

3. $(A \cap B)^c \cap A = A \setminus (A \cap B) = A \setminus B$.
   So $\boxed{A - B}$.

4. Using De Morgan:
   $$A - (B \cap C) = A \cap (B \cap C)^c = A \cap (B^c \cup C^c) = (A - B) \cup (A - C).$$
   So $\boxed{(A - B) \cup (A - C)}$.

5. $A \cup B = \{1, 2, 3, 4\}, \; A \cap B = \{3\}$. Thus
   $$(A \cup B) \times (A \cap B) = \{(1, 3), (2, 3), (3, 3), (4, 3)\}.$$
   Answer: $\boxed{\{(1, 3), (2, 3), (3, 3), (4, 3)\}}$.

6. $A = \{1, 4, 9\}$ has 3 elements. Number of proper subsets $= 2^3 - 1 = 7$.
   Answer: $\boxed{7}$.

7. Which statement is *not* correct? The union of two transitive relations need not be transitive (a simple counterexample exists). Hence the statement "R and S transitive $\implies R \cup S$ transitive" is false.
   So the incorrect statement is option (a).

8. By the general lower bound (inclusion principle), for k properties the minimum percentage losing all k is
   $$\max\big(0, \sum p_i - (k - 1) \cdot 100\big).$$
   Here $\sum p_i = 70 + 80 + 75 + 85 = 310$, $k = 4$, so minimum $= 310 - 3 \cdot 100 = 10$.
   Answer: $\boxed{10\%}$.

9. The first, third and fourth options describe a function/mapping with domain and codomain; the notation "$f : x \mapsto f(x)$" is simply a rule and does not by itself specify domain and codomain. Thus option (b) is the one that is different.

10. The complete solution set of the congruence $x \equiv 3 \pmod 7$ is $\{7p+3 : \ p \in \mathbb{Z}\}$.
So $\boxed{\{7p+3 : p \in \mathbb{Z}\}}$.

11. $|A| = m, \ |B| = n \Rightarrow |A \times B| = mn$.
Answer: $\boxed{mn}$.

12. Of the listed options, only (d) is always true: if R and S are reflexive then $R \cup S$ is reflexive. The other statements (about unions being transitive or unions being equivalence) need not hold.

13. Relation defined by $y = x - 1$. For $x \in A$ we check if $y \in B$:

$$11 \mapsto 10 \ (\in B), \quad 12 \mapsto 11 \ (\notin B), \quad 13 \mapsto 12 \ (\in B).$$

Hence $R = \{(11, 10), (13, 12)\}$.

14. Solve $e^x = e^{-x} \Rightarrow e^{2x} = 1 \Rightarrow x = 0$. Then $y = e^0 = 1$. So $A \cap B = \{(0, 1)\}$ (a singleton).

15. Number of relations from A (size m) to B (size n) is 2^{mn}. Given $2^{mn} = 64 = 2^6$ so $mn = 6$. From the options the pair $(m, n) = (2, 3)$ fits.

16. Let a, b be roots of $4x + \frac{1}{x} = 3$. Put $p = 4x + \frac{1}{x} = 3$. Then

$$p^3 = (4x)^3 + \frac{1}{x^3} + 3 \cdot (4x)^2 \frac{1}{x} + 3 \cdot (4x) \frac{1}{x^2}.$$

Simplifying the mixed terms gives $48x + 12/x = 12(4x + 1/x) = 12p = 36$. Thus

$$p^3 = f(x) + 36, \qquad f(x) = (4x)^3 + \frac{1}{x^3}.$$

With $p = 3$ we get $f(x) = 3^3 - 36 = 27 - 36 = -9$. Hence $f(a) = f(b) = -9$, so option (a) is correct.

17. Linear maps that send the interval $[-1, 1]$ onto $[0, 2]$ must send endpoints to endpoints (either preserving or swapping). Solving the two possibilities yields

$$f(x) = x + 1 \quad \text{and} \quad f(x) = -x + 1.$$

So both maps listed in option (b) are valid.

SET 2 – Comprehension and matrix match (solutions)

Given: For all $x \in \mathbb{R} \setminus \{0, 1\}$,

$$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x, \qquad g(x) = 2f(x) - x + 1.$$

(1) We first solve for f. Let $\phi(x) = \dfrac{x-1}{x}$. Note the three-term system obtained by replacing x by $x,\ \phi(x),\ \phi^2(x)$ gives a linear system for $A = f(x),\ B = f(\phi(x)),\ C = f(\phi^2(x))$:

$$\begin{aligned} A + B &= 1 + x, \\ B + C &= 1 + \phi(x), \\ C + A &= 1 + \phi^2(x). \end{aligned}$$

Solving (add (1)+(3)-(2)) yields

$$f(x) = \tfrac{1}{2}\big(1 + x + \phi^2(x) - \phi(x)\big).$$

Now $\phi(x) = 1 - \dfrac{1}{x}, \quad \phi^2(x) = -\dfrac{1}{x-1}$. Hence

$$f(x) = \tfrac{1}{2}\Big(x + \frac{1}{x} - \frac{1}{x-1}\Big).$$

Therefore

$$g(x) = 2f(x) - x + 1 = 1 + \frac{1}{x} - \frac{1}{x-1} = 1 - \frac{1}{x(x-1)}.$$

The domain of $\sqrt{g(x)}$ requires $g(x) \geq 0$ and $x \neq 0, 1$. Solving

$$1 - \frac{1}{x(x-1)} \geq 0 \iff \frac{x(x-1) - 1}{x(x-1)} \geq 0,$$

which is equivalent to $x(x-1) < 0$ *or* $x(x-1) \geq 1$. Thus

$$x \in (0,1)\ \cup\ \big(-\infty, \tfrac{1-\sqrt{5}}{2}\big] \cup \big[\tfrac{1+\sqrt{5}}{2}, \infty\big), \quad x \neq 0, 1.$$

This set *does not* match any of the four printed options exactly, so the correct choice is: $\boxed{\text{none of the given options}}$.

(2) Range of $g(x)$. Put $y = g(x) = 1 - \dfrac{1}{x(x-1)}$. For $y \neq 1$ we can solve for x; discriminant analysis gives that real solutions exist precisely when $y < 1$ or $y \geq 5$. Hence the range is

$$\boxed{(-\infty, 1)\ \cup\ [5, \infty)}.$$

This corresponds to option (b).

(3) Number of roots of $g(x) = 1$. Setting $g(x) = 1$ gives $1 - \dfrac{1}{x(x-1)} = 1 \Rightarrow \dfrac{1}{x(x-1)} = 0$ which has no solution. Therefore there are $\boxed{0}$ roots.

(4) Matching table (Column I $\leftrightarrow$ Column II):

- (a) $\tan^{-1}\big(\dfrac{2x}{1-x^2}\big) = 2\tan^{-1} x$ holds for $|x| < 1$. So (a) $\mapsto$ (r).
- (b) $\sin^{-1}(\sin x) = x$ and $\sin(\sin^{-1} x) = x$ are both identical on $x \in [-1, 1]$ (note $[-1,1] \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$). So (b) $\mapsto$ (q).
- (c) $\log_{x^2} 25 = \dfrac{\ln 25}{\ln x^2} = \dfrac{\ln 5}{\ln x} = \log_x 5$ for $x > 0, x \neq 1$. The interval given in Column II that fits is $(0, 1)$, so (c) $\mapsto$ (s).

- (d) The pair is identical only at the special points $x = \pm 1$ (principal-value conventions), so (d) $\mapsto$ (p).

(Thus the matches are: (a) r, (b) q, (c) s, (d) p.)

(5) Let $f(x) = 3x^2 - 7x + c$. If the graph of f touches the graph of its inverse f^{-1} at some point a, then

$$f(a) = a \quad \text{and} \quad f'(a) = \pm 1.$$

From $f'(x) = 6x - 7$ the viable solution with $x > 7/6$ is $6a - 7 = 1 \Rightarrow a = \frac{4}{3}$. Using $f(a) = a$ we get

$$3a^2 - 7a + c = a \Rightarrow c = 8a - 3a^2 = \frac{16}{3}.$$

Hence $[c] = \lfloor 16/3 \rfloor = 5$.

(6) Domain of

$$y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$$

requires $1 - x > 0$ ($\Rightarrow x < 1$), $\log_{10}(1-x) \neq 0$ ($\Rightarrow x \neq 0$), and $x + 2 \geq 0$ ($\Rightarrow x \geq -2$). Thus the domain is

$$\boxed{[-2, 1) \setminus \{0\}}.$$

None of the three given printed options matches this exactly, so choose: $\boxed{\text{none of these}}$.

(7) Given $f : (0, \infty) \to \mathbb{R}$ and $F(x) = \int_0^x f(t)\,dt$, and $f(x^2) = x^2(1 + x)$ for $x > 0$. Put $t = x^2$ so $x = \sqrt{t}$ and

$$f(t) = t\big(1 + \sqrt{t}\big) \quad (\text{for } t > 0).$$

Therefore $f(4) = 4(1 + 2) = 12$. (This value is not among the four given choices.)

All steps include necessary reasoning; where an option from the printed list did not match the correct mathematical answer exactly, that discrepancy is noted explicitly.