

SOLUTIONS FOR SET 2

1. **Question:** For the equation $3x^2 + px + 3 = 0, p > 0$ if one of the roots is square of the other then p is equal to

Solution: Let the roots be α and α^2 . The equation is $3x^2 + px + 3 = 0$.

- (i) **Product of Roots:**

$$\alpha \cdot \alpha^2 = \frac{3}{3} = 1$$

$$\alpha^3 = 1$$

Since α is a root of a quadratic equation with real coefficients, α must be real (or one of the complex cube roots of unity, but usually, a real root is intended unless specified). For α to be a real root, $\alpha = 1$.

- (ii) **Sum of Roots:**

$$\alpha + \alpha^2 = -\frac{p}{3}$$

Substitute $\alpha = 1$:

$$1 + 1^2 = -\frac{p}{3}$$

$$2 = -\frac{p}{3}$$

$$p = -6$$

- (iii) **Complex Roots Case:** If α is a complex root, $\alpha = \omega$ or $\alpha = \omega^2$. Since the coefficients are real, if α is a root, $\bar{\alpha}$ is also a root. If $\alpha = \omega$, the roots are ω and ω^2 .

$$\omega + \omega^2 = -1$$

$$\omega + \omega^2 = -\frac{p}{3}$$

$$-1 = -\frac{p}{3} \implies p = 3$$

Given $p > 0$, $p = 3$ is the correct value. The cubic equation $x^3 = 1$ having complex roots ω, ω^2 is consistent with the quadratic $3x^2 + px + 3 = 0$ having real coefficients. The roots are ω and ω^2 .

Answer: (a) 3

2. **Question:** If $\alpha, \beta (\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$ where $c < 0 < b$ then

Solution: The equation is $x^2 + bx + c = 0$.

- (i) **Product of Roots ($\alpha\beta$):**

$$\alpha\beta = c$$

Since $c < 0$, the product $\alpha\beta$ is negative. This means the two roots, α and β , must have ****opposite signs****.

- (ii) **Sum of Roots ($\alpha + \beta$):**

$$\alpha + \beta = -b$$

Since $b > 0$, the sum $\alpha + \beta$ is negative ($\alpha + \beta < 0$).

- (iii) **Conclusion on Roots:** Since the roots have opposite signs, one is positive and one is negative. Since $\alpha < \beta$, we must have $\alpha < 0 < \beta$.

- (iv) **Comparing Magnitudes:** The sum is $\alpha + \beta < 0$. Since α is negative and β is positive:

$$\alpha + \beta < 0 \implies \beta < -\alpha$$

Since $\alpha < 0$, $-\alpha = |\alpha|$.

$$\beta < |\alpha|$$

- (v) Combining all conditions: $\alpha < 0 < \beta < |\alpha|$.

Answer: (a) $\alpha < 0 < \beta < |\alpha|$

3. **Question:** If $b > a$ then the equation $(x - a)(x - b) = 1$ has

Solution: The equation is $(x - a)(x - b) - 1 = 0$. Let $f(x) = (x - a)(x - b) - 1$. The roots of $f(x) = 0$ are the points where the graph of the parabola $y = (x - a)(x - b)$ is 1 unit above the x-axis.

- (i) $f(x)$ is a quadratic in x with a positive leading coefficient (the coefficient of x^2 is 1), so the parabola opens ****upwards****.

(ii) Evaluate $f(x)$ at the critical points $x = a$ and $x = b$:

$$f(a) = (a - a)(a - b) - 1 = 0 - 1 = -1$$

$$f(b) = (b - a)(b - b) - 1 = 0 - 1 = -1$$

(iii) Since $f(a) = -1 < 0$ and $f(b) = -1 < 0$, the roots of $f(x) = 0$ cannot lie in the interval $[a, b]$, because the vertex of the parabola lies between a and b , and at a and b the function is already negative. Since the parabola opens upward, $f(x)$ cannot cross the x-axis between a and b .

(iv) Now check the limits:

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} x^2 = +\infty$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} x^2 = +\infty$$

(v) Apply the Intermediate Value Theorem:

- In $(-\infty, a)$: $f(x)$ goes from $+\infty$ to $f(a) = -1$. Since the sign changes, there must be **one root in $(-\infty, a)$ **.
- In (b, ∞) : $f(x)$ goes from $f(b) = -1$ to $+\infty$. Since the sign changes, there must be **one root in (b, ∞) **.

Answer: (c) one root in $(-\infty, a)$ and the other roots in (b, ∞)

4. **Question:** The number of solution of the equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ is

Solution: The equation is $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$.

(i) **Domain:** For the square roots to be defined, the terms inside must be non-negative:

- $x+1 \geq 0 \implies x \geq -1$
- $x-1 \geq 0 \implies x \geq 1$
- $4x-1 \geq 0 \implies x \geq 1/4$

The intersection of these conditions is the domain: $x \geq 1$.

(ii) **Analysis of signs:** For $x \geq 1$:

- $\sqrt{x+1} > 0$
- $\sqrt{x-1} \geq 0$
- $\sqrt{4x-1} > 0$

Also, for $x \geq 1$, $x+1 > x-1$, so $\sqrt{x+1} > \sqrt{x-1}$. The LHS, $\sqrt{x+1} - \sqrt{x-1}$, is ≥ 0 .

(iii) **Solve the equation:** Square both sides:

$$(\sqrt{x+1} - \sqrt{x-1})^2 = 4x-1$$

$$(x+1) + (x-1) - 2\sqrt{(x+1)(x-1)} = 4x-1$$

$$2x - 2\sqrt{x^2-1} = 4x-1$$

$$1 - 2x = 2\sqrt{x^2-1}$$

(iv) For the last equation to have a solution, the LHS must be non-negative:

$$1 - 2x \geq 0 \implies 2x \leq 1 \implies x \leq 1/2$$

(v) **Contradiction:** We established that any solution must satisfy the domain $x \geq 1$ (from step i) and the algebraic constraint $x \leq 1/2$ (from step iv). Since these two conditions contradict each other, there are **no solutions**.

Answer: (a) 0

5. **Question:** The value of a for which the equation $2x^2 + 2\sqrt{6}x + a = 0$ has equal roots is

Solution: For a quadratic equation $Ax^2 + Bx + C = 0$ to have equal roots, the discriminant D must be zero.

$$D = B^2 - 4AC = 0$$

(i) Identify coefficients: $A = 2, B = 2\sqrt{6}, C = a$.

(ii) Set $D = 0$:

$$(2\sqrt{6})^2 - 4(2)(a) = 0$$

(iii) Solve for a :

$$4(6) - 8a = 0$$

$$24 - 8a = 0$$

$$8a = 24$$

$$a = \frac{24}{8} = \mathbf{3}$$

Answer: (a) 3

6. **Question:** The solution of the equation $(3 + 2\sqrt{2})^{x^2-8} + (3 + 2\sqrt{2})^{8-x^2} = 6$ are

Solution: Let $y = x^2 - 8$. The equation is $(3 + 2\sqrt{2})^y + (3 + 2\sqrt{2})^{-y} = 6$.

(i) Notice that $3 + 2\sqrt{2}$ and $3 - 2\sqrt{2}$ are reciprocals:

$$(3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 3^2 - (2\sqrt{2})^2 = 9 - 8 = 1$$

$$\text{So, } \frac{1}{3 + 2\sqrt{2}} = 3 - 2\sqrt{2}.$$

(ii) Rewrite the equation using $3 - 2\sqrt{2} = (3 + 2\sqrt{2})^{-1}$:

$$(3 + 2\sqrt{2})^{x^2-8} + (3 - 2\sqrt{2})^{x^2-8} = 6$$

(iii) Let $t = (3 + 2\sqrt{2})^{x^2-8}$. Since $3 + 2\sqrt{2} > 0$, $t > 0$. The equation becomes:

$$t + \frac{1}{t} = 6$$

(iv) Solve the quadratic equation in t :

$$t^2 + 1 = 6t$$

$$t^2 - 6t + 1 = 0$$

Using the quadratic formula for t :

$$t = \frac{6 \pm \sqrt{36 - 4(1)(1)}}{2} = \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$

(v) Substitute back $t = (3 + 2\sqrt{2})^{x^2-8}$:

$$\text{Case 1: } (3 + 2\sqrt{2})^{x^2-8} = 3 + 2\sqrt{2}$$

$$(3 + 2\sqrt{2})^{x^2-8} = (3 + 2\sqrt{2})^1$$

$$x^2 - 8 = 1 \implies x^2 = 9 \implies \mathbf{x = \pm 3}$$

$$\text{Case 2: } (3 + 2\sqrt{2})^{x^2-8} = 3 - 2\sqrt{2}$$

$$(3 + 2\sqrt{2})^{x^2-8} = (3 + 2\sqrt{2})^{-1}$$

$$x^2 - 8 = -1 \implies x^2 = 7 \implies \mathbf{x = \pm\sqrt{7}}$$

(vi) The solutions are $x = \pm 3, \pm\sqrt{7}$.

Answer: (e) $\pm 3, \pm\sqrt{7}$

7. **Question:** If a, b, c are the sides of a triangle ABC such that $a \neq b \neq c$ and $x^2 - 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ has real roots then

Solution: For the quadratic equation $Ax^2 + Bx + C = 0$ to have real roots, the discriminant D must be non-negative ($D \geq 0$).

(i) Identify coefficients: $A = 1, B = -2(a + b + c), C = 3\lambda(ab + bc + ca)$.

$$D = B^2 - 4AC = [-2(a + b + c)]^2 - 4(1)[3\lambda(ab + bc + ca)]$$

$$D = 4(a + b + c)^2 - 12\lambda(ab + bc + ca)$$

(ii) Set $D \geq 0$:

$$4(a+b+c)^2 - 12\lambda(ab+bc+ca) \geq 0$$

$$4(a^2+b^2+c^2+2ab+2bc+2ca) - 12\lambda(ab+bc+ca) \geq 0$$

Divide by 4:

$$a^2+b^2+c^2+2(ab+bc+ca) - 3\lambda(ab+bc+ca) \geq 0$$

$$a^2+b^2+c^2 \geq (3\lambda-2)(ab+bc+ca)$$

$$\lambda \leq \frac{a^2+b^2+c^2+2(ab+bc+ca)}{3(ab+bc+ca)}$$

(iii) We use the identity $a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$. From (ii):

$$a^2+b^2+c^2 - (3\lambda-2)(ab+bc+ca) \geq 0$$

Substitute $a^2+b^2+c^2 = ab+bc+ca + \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$:

$$ab+bc+ca + \frac{1}{2} \sum (a-b)^2 - (3\lambda-2)(ab+bc+ca) \geq 0$$

$$(ab+bc+ca)(1-(3\lambda-2)) + \frac{1}{2} \sum (a-b)^2 \geq 0$$

$$(ab+bc+ca)(3-3\lambda) + \frac{1}{2} \sum (a-b)^2 \geq 0$$

$$3(1-\lambda)(ab+bc+ca) + \frac{1}{2} \sum (a-b)^2 \geq 0$$

(iv) Since a, b, c are sides of a triangle, $a, b, c > 0$, so $ab+bc+ca > 0$. Since $a \neq b \neq c$, we have $\sum (a-b)^2 > 0$.

(v) We use the known inequality $a^2+b^2+c^2 \geq ab+bc+ca$. From (ii), we need $a^2+b^2+c^2+2(ab+bc+ca) \geq 3\lambda(ab+bc+ca)$. Since $a^2+b^2+c^2 \geq ab+bc+ca$, we have:

$$3(ab+bc+ca) \leq a^2+b^2+c^2+2(ab+bc+ca)$$

So, the maximum value of $3\lambda(ab+bc+ca)$ is bounded by $a^2+b^2+c^2+2(ab+bc+ca)$.

$$\lambda \leq \frac{a^2+b^2+c^2}{3(ab+bc+ca)} + \frac{2}{3}$$

(vi) Since $a \neq b \neq c$, $\frac{a^2+b^2+c^2}{ab+bc+ca} > 1$. The minimum value of $\frac{a^2+b^2+c^2}{ab+bc+ca}$ is strictly greater than 1 (it equals 1 only if $a=b=c$).

$$\lambda < \frac{1}{3} \cdot \frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{2}{3}$$

The inequality is $\lambda \leq 1 + \frac{(a-b)^2+(b-c)^2+(c-a)^2}{6(ab+bc+ca)}$. Since $a \neq b \neq c$, the fraction is > 0 . Thus, λ must be

< 1 . The inequality must be $3(1-\lambda) \geq 0$ (approximate) $\implies \lambda \leq 1$.

If $a=b=c$, then $\lambda \leq 1$. Since $a \neq b \neq c$, $\lambda < 1$. The options suggest a lower bound $\lambda < 4/3$ or an interval. Let's recheck the inequality:

$$3\lambda \leq \frac{a^2+b^2+c^2}{ab+bc+ca} + 2$$

Since a, b, c form a triangle, the sides are positive. There is no special known bound for $\frac{a^2+b^2+c^2}{ab+bc+ca}$ other than ≥ 1 .

Let's check the options. If we use the Cauchy-Schwarz inequality, $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. Let $a^2+b^2+c^2 = k(ab+bc+ca)$. $k > 1$. $D = 4(k(ab+bc+ca) + 2(ab+bc+ca)) - 12\lambda(ab+bc+ca) \geq 0$
 $4(k+2) \geq 12\lambda \implies \lambda \leq \frac{k+2}{3}$. Since $k > 1$, $\lambda < \frac{\infty+2}{3} = \infty$.

Since a, b, c are sides, they are positive. Since $a \neq b \neq c$, λ can be close to 1 but must be < 1 if the problem implied $\lambda \leq 1$. The only option that makes sense is λ being bounded by a value > 1 . There is likely an error in the provided options or problem statement as typically $D \geq 0$ yields $\lambda \leq 1$.

Assuming $\lambda < 4/3$ is the intended answer (perhaps from a different constraint):

$$3\lambda(ab+bc+ca) \leq (a+b+c)^2 \quad \text{must hold.}$$

The strongest condition we can impose is $\lambda \leq 1$. Given the options, (a) $\lambda < 4/3$ is the largest upper bound provided, which is true if $\lambda \leq 1$. We assume (a) is the intended answer.

Answer: (a) $\lambda < \frac{4}{3}$ (Based on $D \geq 0$ leading to $\lambda \leq 1$, this condition is satisfied).

8. **Question:** If $e^{\cos x} - e^{-\cos x} = 4$ then the value of the $\cos x$ is

Solution: Let $y = \cos x$. Since $-1 \leq \cos x \leq 1$, we have $e^{-1} \leq e^y \leq e^1$. The equation is $e^y - e^{-y} = 4$.

(i) Let $t = e^y$. Since y is real, $t > 0$. The equation is:

$$t - \frac{1}{t} = 4$$

(ii) Solve the quadratic equation in t :

$$t^2 - 1 = 4t$$

$$t^2 - 4t - 1 = 0$$

$$t = \frac{4 \pm \sqrt{16 - 4(1)(-1)}}{2} = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$

(iii) Since $t = e^y > 0$, we must choose the positive root:

$$t = 2 + \sqrt{5} \quad (\text{since } 2 - \sqrt{5} < 0)$$

(iv) Substitute back $t = e^y$ and solve for y :

$$e^y = 2 + \sqrt{5}$$

$$y = \ln(2 + \sqrt{5})$$

(v) Since $y = \cos x$:

$$\cos x = \ln(2 + \sqrt{5})$$

(vi) Check for consistency: $\cos x$ must be in $[-1, 1]$. Since $\sqrt{5} \approx 2.236$, $2 + \sqrt{5} \approx 4.236$. $\ln(4.236) \approx 1.44$. Since $1.44 > 1$, $\cos x$ is outside the range $[-1, 1]$.

(vii) Since there is no real value of x for which $e^{\cos x} - e^{-\cos x} = 4$, the value of $\cos x$ is not attainable for any real x . However, the question asks for the value of $\cos x$ itself.

$$\ln(2 + \sqrt{5}) = \log_e(2 + \sqrt{5})$$

Using base e , this matches option (a) if $\log$ is assumed to be $\ln$.

Answer: (a) $\log(2 + \sqrt{5})$ (Assuming $\log = \ln$; note that this value is > 1 , so no real x exists).

9. **Question:** $\sin A, \sin B, \cos A$ are in GP, Roots of $x^2 + 2x \cot B + 1 = 0$ are always

Solution:

(i) If $\sin A, \sin B, \cos A$ are in G.P., then:

$$(\sin B)^2 = (\sin A)(\cos A)$$

$$\sin^2 B = \frac{1}{2}(2 \sin A \cos A) = \frac{1}{2} \sin(2A)$$

$$\sin(2A) = 2 \sin^2 B$$

(ii) Consider the discriminant D of the quadratic equation $x^2 + 2x \cot B + 1 = 0$:

$$D = (2 \cot B)^2 - 4(1)(1) = 4 \cot^2 B - 4 = 4(\cot^2 B - 1)$$

(iii) The nature of the roots depends on the sign of D .

- If $\cot^2 B - 1 > 0 \implies \cot^2 B > 1$: $D > 0$, roots are real and distinct.
- If $\cot^2 B - 1 = 0 \implies \cot^2 B = 1$: $D = 0$, roots are real and equal.
- If $\cot^2 B - 1 < 0 \implies \cot^2 B < 1$: $D < 0$, roots are imaginary.

(iv) We analyze the given condition $\sin(2A) = 2 \sin^2 B$. Since A is an angle, $-1 \leq \sin(2A) \leq 1$. Since B is an angle, $0 \leq \sin^2 B \leq 1$.

$$2 \sin^2 B = \sin(2A)$$

This implies $0 \leq 2 \sin^2 B \leq 1$, so $0 \leq \sin^2 B \leq 1/2$.

(v) If $0 \leq \sin^2 B \leq 1/2$, then:

$$\frac{1}{\sin^2 B} \geq 2$$

$$\csc^2 B \geq 2$$

(vi) Use the identity $\csc^2 B = 1 + \cot^2 B$:

$$1 + \cot^2 B \geq 2$$

$$\cot^2 B \geq 1$$

(vii) Since $\cot^2 B \geq 1$, the discriminant $D = 4(\cot^2 B - 1)$ is $D \geq 0$.

(viii) Therefore, the roots are always **real** (either real and distinct, or real and equal).

Answer: (a) real

10. **Question:** If $a = \log_2 3, b = \log_2 5, c = \log_7 2$ then $\log_{140} 63$ in terms of a, b, c is

Solution: Given: $a = \log_2 3, b = \log_2 5, c = \log_7 2$. We want to find $L = \log_{140} 63$. Use the change of base formula to base 2.

(i) Express c in base 2:

$$c = \log_7 2 = \frac{1}{\log_2 7} \implies \log_2 7 = \frac{1}{c}$$

(ii) Express L in base 2:

$$L = \log_{140} 63 = \frac{\log_2 63}{\log_2 140}$$

(iii) Factor the numbers: $63 = 9 \times 7 = 3^2 \times 7$ and $140 = 14 \times 10 = 2 \times 7 \times 2 \times 5 = 2^2 \times 5 \times 7$.

(iv) Simplify the numerator $\log_2 63$:

$$\log_2 63 = \log_2(3^2 \cdot 7) = 2\log_2 3 + \log_2 7 = 2a + \frac{1}{c}$$

(v) Simplify the denominator $\log_2 140$:

$$\log_2 140 = \log_2(2^2 \cdot 5 \cdot 7) = \log_2 2^2 + \log_2 5 + \log_2 7 = 2 + b + \frac{1}{c}$$

(vi) Substitute back into L :

$$L = \frac{2a + \frac{1}{c}}{2 + b + \frac{1}{c}}$$

(vii) Multiply numerator and denominator by c :

$$L = \frac{c(2a) + c(\frac{1}{c})}{c(2) + c(b) + c(\frac{1}{c})} = \frac{2ac + 1}{2c + bc + 1}$$

Answer: (a) $\frac{2ac + 1}{2c + abc + 1}$ (Note: Option (a) has abc instead of bc . We assume this is a typo in the option and bc is correct, as $\log_2 5 \cdot \log_7 2 = b \cdot c = \log_7 5$). The correct answer is $\frac{2ac + 1}{2c + bc + 1}$. Since a, b, c are factors, $bc = abc/a$. Option (a) is the closest form provided. We assume bc was intended.

11. **Question:** If α and β are the roots of the equation $ax^2 + bx + c = 0$ ($c \neq 0$) then equation whose roots are $\frac{1}{a\alpha + b}$ and $\frac{1}{a\beta + b}$ is

Solution: The equation is $ax^2 + bx + c = 0$. Since α is a root, $a\alpha^2 + b\alpha + c = 0$.

(i) From $a\alpha^2 + b\alpha + c = 0$:

$$\begin{aligned} a\alpha^2 + b\alpha &= -c \\ \alpha(a\alpha + b) &= -c \implies a\alpha + b = -\frac{c}{\alpha} \end{aligned}$$

(ii) Similarly, since β is a root:

$$a\beta + b = -\frac{c}{\beta}$$

(iii) The new roots, y_1 and y_2 , are:

$$\begin{aligned} y_1 &= \frac{1}{a\alpha + b} = \frac{1}{-c/\alpha} = -\frac{\alpha}{c} \\ y_2 &= \frac{1}{a\beta + b} = \frac{1}{-c/\beta} = -\frac{\beta}{c} \end{aligned}$$

(iv) The required equation has roots $y_1 = -\frac{\alpha}{c}$ and $y_2 = -\frac{\beta}{c}$.

(v) **Sum of new roots (S'):**

$$S' = y_1 + y_2 = -\frac{\alpha}{c} - \frac{\beta}{c} = -\frac{1}{c}(\alpha + \beta)$$

From the original equation: $\alpha + \beta = -b/a$.

$$S' = -\frac{1}{c} \left(-\frac{b}{a} \right) = \frac{b}{ac}$$

(vi) **Product of new roots (P'):**

$$P' = y_1 y_2 = \left(-\frac{\alpha}{c} \right) \left(-\frac{\beta}{c} \right) = \frac{\alpha\beta}{c^2}$$

From the original equation: $\alpha\beta = c/a$.

$$P' = \frac{c/a}{c^2} = \frac{c}{ac^2} = \frac{1}{ac}$$

(vii) The new quadratic equation is $x^2 - S'x + P' = 0$:

$$x^2 - \left(\frac{b}{ac} \right) x + \frac{1}{ac} = 0$$

(viii) Multiply by ac to clear denominators:

$$acx^2 - bx + 1 = 0$$

Answer: (a) $acx^2 - bx + 1 = 0$

12. **Question:** If α, β be the roots of $x^2 - a(x-1) + b = 0$ then the value of $\frac{1}{\alpha^2 - a\alpha} + \frac{1}{\beta^2 - a\beta} + \frac{2}{a+b}$ is

Solution: The equation is $x^2 - ax + a + b = 0$. Since α is a root:

$$\alpha^2 - a\alpha + a + b = 0 \implies \alpha^2 - a\alpha = -(a+b)$$

Similarly, since β is a root:

$$\beta^2 - a\beta = -(a+b)$$

(i) Substitute these values into the expression:

$$E = \frac{1}{\alpha^2 - a\alpha} + \frac{1}{\beta^2 - a\beta} + \frac{2}{a+b}$$

$$E = \frac{1}{-(a+b)} + \frac{1}{-(a+b)} + \frac{2}{a+b}$$

$$E = -\frac{1}{a+b} - \frac{1}{a+b} + \frac{2}{a+b}$$

(ii) Combine the terms:

$$E = \frac{-1-1+2}{a+b} = \frac{0}{a+b} = 0$$

Answer: (c) 0

13. **Question:** The number of real roots of the equation $x^4 + \sqrt{x^4 + 20} = 22$ is

Solution: The equation is $x^4 + \sqrt{x^4 + 20} = 22$.

(i) **Domain:** The term inside the square root must be non-negative: $x^4 + 20 \geq 0$. Since $x^4 \geq 0$, this is always true for all real x .

(ii) **Substitution:** Let $y = x^4 \geq 0$. The equation becomes:

$$y + \sqrt{y + 20} = 22$$

(iii) Isolate the square root:

$$\sqrt{y + 20} = 22 - y$$

(iv) **Constraint:** Since the LHS is ≥ 0 , we must have $22 - y \geq 0 \implies y \leq 22$. Since $y = x^4$, we have the constraint $0 \leq y \leq 22$.

(v) **Square both sides:**

$$\begin{aligned}y + 20 &= (22 - y)^2 \\y + 20 &= 484 - 44y + y^2 \\y^2 - 45y + 464 &= 0\end{aligned}$$

(vi) **Solve the quadratic equation for y :** Using the quadratic formula:

$$\begin{aligned}y &= \frac{-(-45) \pm \sqrt{(-45)^2 - 4(1)(464)}}{2} \\y &= \frac{45 \pm \sqrt{2025 - 1856}}{2} = \frac{45 \pm \sqrt{169}}{2} = \frac{45 \pm 13}{2}\end{aligned}$$

Two possible values for y :

$$\begin{aligned}y_1 &= \frac{45 + 13}{2} = \frac{58}{2} = 29 \\y_2 &= \frac{45 - 13}{2} = \frac{32}{2} = 16\end{aligned}$$

(vii) **Check the Constraint $0 \leq y \leq 22$:**

- $y_1 = 29$: This is outside the constraint $y \leq 22$. **Reject.**
- $y_2 = 16$: This is within the constraint $0 \leq 16 \leq 22$. **Accept.**

(viii) **Solve for x :** $y = x^4$, so $x^4 = 16$.

$$x = \pm \sqrt[4]{16} = \pm 2$$

(ix) There are **two** real roots, $x = 2$ and $x = -2$.

Answer: (b) 2

14. **Question:** Let a, b, c be real if $ax^2 + bx + c = 0$ has two real roots α, β where $\alpha < -2$ and $\beta > 2$ then

Solution: Let $f(x) = ax^2 + bx + c$. The roots are α and β such that $\alpha < -2$ and $\beta > 2$. The conditions imply that the interval $[-2, 2]$ lies strictly between the roots.

- (i) The condition for a number k to lie between the roots is $a \cdot f(k) < 0$.
- (ii) Since both -2 and 2 lie between the roots α and β , the value of $f(x)$ at $x = -2$ and $x = 2$ must have the opposite sign to the leading coefficient a .
- (iii) Condition for $x = -2$:

$$\begin{aligned}a \cdot f(-2) &< 0 \\f(-2) &= a(-2)^2 + b(-2) + c = 4a - 2b + c \\a(4a - 2b + c) &< 0\end{aligned}$$

Divide by a^2 (since $a \neq 0$ for a quadratic):

$$\begin{aligned}\frac{1}{a}(4a - 2b + c) &< 0 \\4 - \frac{2b}{a} + \frac{c}{a} &< 0\end{aligned}$$

(iv) Condition for $x = 2$:

$$\begin{aligned}a \cdot f(2) &< 0 \\f(2) &= a(2)^2 + b(2) + c = 4a + 2b + c \\a(4a + 2b + c) &< 0\end{aligned}$$

Divide by a :

$$4 + \frac{2b}{a} + \frac{c}{a} < 0$$

(v) Both $4 - \frac{2b}{a} + \frac{c}{a} < 0$ and $4 + \frac{2b}{a} + \frac{c}{a} < 0$ are necessarily true.

(vi) Option (a) is $4 - \frac{2b}{a} + \frac{c}{a} < 0$, which is the condition derived from $x = -2$.

(vii) Option (b) is $4 + \frac{2b}{a} + \frac{c}{a} < 0$. This is $4 + \frac{2b}{a} + \frac{c}{a} < 2\frac{c}{a}$. This is not one of the derived conditions.

The derived conditions are $4 - \frac{2b}{a} + \frac{c}{a} < 0$ and $4 + \frac{2b}{a} + \frac{c}{a} < 0$. Option (a) is one of the necessary conditions.

Answer: (a) $4 - \frac{2b}{a} + \frac{c}{a} < 0$