SOLUTIONS TO COMPLEX NUMBERS (SET 2)

1. Let z and w be two complex numbers such that $|z| = |w|$ and $\arg z + \arg w = \pi$, then z equals :
   A. w B. -w C. $\overline{w}$ D. $-\overline{w}$

   Solution: Let $z = re^{i\theta_z}$ and $w = re^{i\theta_w}$, where $r = |z| = |w|$. The second condition is $\theta_z + \theta_w = \pi$. Thus, $\theta_z = \pi - \theta_w$.

   Substitute θ_z into the expression for z:

$$\begin{aligned} z &= re^{i(\pi-\theta_w)} \\ &= r(\cos(\pi - \theta_w) + i\sin(\pi - \theta_w)) \\ &= r(-\cos\theta_w + i\sin\theta_w) \end{aligned}$$

   Now consider $-\overline{w}$:

$$\overline{w} = re^{-i\theta_w} = r(\cos\theta_w - i\sin\theta_w)$$

$$-\overline{w} = -r(\cos\theta_w - i\sin\theta_w) = r(-\cos\theta_w + i\sin\theta_w)$$

   Comparing the two results, we find $z = -\overline{w}$.

   Answer: $-\overline{w}$

2. Let z and w be two complex numbers such that $|z| \leq 1, |w| \leq 1$ and $|z + iw| = |z - i\overline{w}| = 2$, then z equals :
   A. 1 or i B. i or -i C. 1 or -1 D. i or -1

   Solution: We are given $|z| \leq 1$ and $|w| \leq 1$.

   The Triangle Inequality states $|z_1 + z_2| \leq |z_1| + |z_2|$. Since $|z| \leq 1$ and $|iw| = |i||w| = |w| \leq 1$, we have:

$$|z + iw| \leq |z| + |iw| \leq 1 + 1 = 2$$

   For $|z + iw| = 2$, the equality must hold:

$$|z + iw| = |z| + |iw|$$

   This implies that z and iw must be collinear with the origin, and $z = \lambda(iw)$ for some $\lambda > 0$. Since $|z| \leq 1$ and $|iw| \leq 1$, and $|z + iw| = 2$, we must have $|z| = 1$ and $|iw| = 1$, so $|w| = 1$.

   Also, $\arg(z) = \arg(iw) \implies \arg(z) = \arg(i) + \arg(w) = \frac{\pi}{2} + \arg(w)$.

   From $|z - i\overline{w}| = 2$, similarly, we must have $|z| = 1$ and $|-i\overline{w}| = 1$, which means $|w| = 1$.

$$|z| + |-i\overline{w}| = |z - i\overline{w}|$$

   This implies $z = \mu(-i\overline{w})$ for some $\mu > 0$. Since $|z| = 1$ and $|-i\overline{w}| = 1$, we must have $\mu = 1$, so $z = -i\overline{w}$.

   Let $w = \cos\theta + i\sin\theta$ (since $|w| = 1$). Then $\overline{w} = \cos\theta - i\sin\theta$.

$$z = -i\overline{w} = -i(\cos\theta - i\sin\theta) = -i\cos\theta - i^2\sin\theta = \sin\theta - i\cos\theta$$

   Also, $|z| = 1$, so $z = \cos\phi + i\sin\phi$.

   Comparing $\sin\theta - i\cos\theta$ with $\cos\phi + i\sin\phi$:

$$\cos\phi = \sin\theta \quad \text{and} \quad \sin\phi = -\cos\theta$$

   From $\cos\phi = \sin\theta$ and $\sin\phi = -\cos\theta$, we see $\phi = \theta - \frac{\pi}{2}$.

   Now we use the first condition $z = iw$:

$$z + iw = 2 \implies z = -iw \quad \text{with a correction on the previous step:}$$

   The condition for equality in triangle inequality is $\frac{z}{|z|} = \frac{iw}{|iw|}$. Since $|z| = 1$ and $|iw| = 1$, we have $\mathbf{z = iw}$.

Substitute $z = -i\overline{w}$ into $z = iw$:

$$-i\overline{w} = iw$$

$$\overline{w} = -w$$

If $w = x + iy$, then $\overline{w} = x - iy$.

$$x - iy = -(x + iy) = -x - iy$$

$$x = -x \implies 2x = 0 \implies x = 0$$

So w must be purely imaginary, $w = iy$, with $|w| = 1 \implies |iy| = 1 \implies |y| = 1$.

$$w = i \quad \text{or} \quad w = -i$$

Case 1: $w = i$.

$$z = iw = i(i) = i^2 = -1$$

Case 2: $w = -i$.

$$z = iw = i(-i) = -i^2 = 1$$

Thus, $z = 1$ or $z = -1$.

Answer: 1 or -1

3. For positive integers n_1, n_2 the value of expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, here $i = \sqrt{-1}$ is a real number, if and only if :
A. $n_1 = n_2 + 1$ B. $n_1 = n_2 - 1$ C. $n_1 = n_2$ D. $n_1 > 0, n_2 > 0$

Solution: First, simplify the terms inside the parentheses using the properties of i: $i^2 = -1, i^3 = -i, i^4 = 1$.

- $1 + i^3 = 1 - i$
- $1 + i^5 = 1 + i^4 \cdot i = 1 + i$
- $1 + i^7 = 1 + i^4 \cdot i^3 = 1 + i^3 = 1 - i$

Let E be the expression:

$$E = (1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2}$$

E is real if and only if $\text{Im}(E) = 0$. Since the sum of a complex number and its conjugate is real, the sum of conjugates is always real:

- $(1+i)^{n_1} + (1-i)^{n_1}$ is real, since $(1-i) = \overline{(1+i)}$ and $(1-i)^{n_1} = \overline{(1+i)^{n_1}}$.
- Similarly, $(1+i)^{n_2} + (1-i)^{n_2}$ is real.

Since both parts are always real for any positive integers n_1, n_2, the entire expression E is always real.

Thus, E is a real number for all positive integers n_1 and n_2.

Answer: $n_1 > 0, n_2 > 0$

4. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ is equal to :
A. 128ω B. -128ω C. $128\omega^2$ D. $-128\omega^2$

Solution: Since ω is a cube root of unity and $\omega \neq 1$, we have the property:

$$1 + \omega + \omega^2 = 0$$

From this, we can replace the term $1 + \omega$:

$$1 + \omega = -\omega^2$$

Substitute this into the expression:

$$\begin{aligned}(1+\omega-\omega^2)^7 &= ((-\omega^2)-\omega^2)^7\\ &= (-2\omega^2)^7\\ &= (-2)^7(\omega^2)^7\\ &= -128\omega^{14}\end{aligned}$$

Using the property $\omega^3 = 1$:

$$\omega^{14} = \omega^{12}\cdot\omega^2 = (\omega^3)^4\cdot\omega^2 = 1^4\cdot\omega^2 = \omega^2$$

So, the expression equals:

$$-128\omega^2$$

Answer: $-128\omega^2$

5. The value of sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$ equals : A. i B. i-1 C. -i D. 0

Solution: The term inside the summation can be factored:

$$i^n + i^{n+1} = i^n(1+i)$$

The sum is:

$$S = \sum_{n=1}^{13} i^n(1+i) = (1+i)\sum_{n=1}^{13} i^n$$

The sum of powers of i over a complete cycle of 4 is zero: $i^1 + i^2 + i^3 + i^4 = i - 1 - i + 1 = 0$. The sum $\sum_{n=1}^{13} i^n$ has 13 terms. It contains $13/4 = 3$ full cycles and 1 remaining term (i^{13}):

$$\begin{aligned}\sum_{n=1}^{13} i^n &= (i^1+i^2+i^3+i^4) + (i^5+\cdots+i^8) + (i^9+\cdots+i^{12}) + i^{13}\\ &= 0+0+0+i^{13}\\ &= i^{4\cdot3+1} = (i^4)^3\cdot i^1 = 1^3\cdot i = i\end{aligned}$$

Substitute this back into the expression for S:

$$S = (1+i)\cdot i = i + i^2 = i - 1$$

Answer: i-1

6. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then : A. x=3,y=1 B. x=1,y=1 C. x=0,y=3 D. x=0,y=0

Solution: Let D be the determinant. Expand the determinant along the first row:

$$\begin{aligned}D &= 6i\begin{vmatrix} 3i & -1 \\ 3 & i \end{vmatrix} - (-3i)\begin{vmatrix} 4 & -1 \\ 20 & i \end{vmatrix} + 1\begin{vmatrix} 4 & 3i \\ 20 & 3 \end{vmatrix}\\ &= 6i((3i)(i) - (-1)(3)) + 3i((4)(i) - (-1)(20)) + 1((4)(3) - (3i)(20))\\ &= 6i(3i^2+3) + 3i(4i+20) + (12-60i)\\ &= 6i(-3+3) + 12i^2 + 60i + 12 - 60i\\ &= 6i(0) - 12 + 60i + 12 - 60i\\ &= 0 - 12 + 12 + 60i - 60i\\ &= 0\end{aligned}$$

Thus, $x + iy = 0$. Since x and y are real, $x = 0$ and $y = 0$.

Answer: x=0,y=0

7. If i $=\sqrt{-1}$, then $4+5(-\frac{1}{2}+\frac{i\sqrt{3}}{2})^{334}+3(-\frac{1}{2}+\frac{i\sqrt{3}}{2})^{365}$ is equal to : A. $1-i\sqrt{3}$ B. $-1+i\sqrt{3}$ C. $i\sqrt{3}$ D. $-i\sqrt{3}$

   Solution: Let ω be the imaginary cube root of unity: $\omega = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$.

   The expression is $E = 4+5\omega^{334}+3\omega^{365}$.

   We use the property $\omega^3 = 1$.

   - $\omega^{334} = \omega^{333}\cdot\omega^1 = (\omega^3)^{111}\cdot\omega = 1^{111}\cdot\omega = \omega$
   - $\omega^{365} = \omega^{363}\cdot\omega^2 = (\omega^3)^{121}\cdot\omega^2 = 1^{121}\cdot\omega^2 = \omega^2$

   Substitute these back into E:

   $$E = 4+5\omega+3\omega^2$$

   We use the property $1+\omega+\omega^2 = 0 \implies \omega^2 = -1-\omega$.

   $$\begin{aligned} E &= 4+5\omega+3(-1-\omega) \\ &= 4+5\omega-3-3\omega \\ &= (4-3)+(5\omega-3\omega) \\ &= 1+2\omega \end{aligned}$$

   Substitute $\omega = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$:

   $$E = 1+2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 1-1+i\sqrt{3} = i\sqrt{3}$$

   Answer: $i\sqrt{3}$

8. If $\arg(z) < 0$, then $\arg(-z)-\arg(z) =$ A. π B. $-\pi$ C. $-\frac{\pi}{2}$ D. $\frac{\pi}{2}$

   Solution: We use the property:

   $$\arg(z_1 z_2) = \arg(z_1)+\arg(z_2) \pmod{2\pi}$$

   Let $z_1 = -1$. Then $-z = (-1)z$.

   $$\arg(-z) = \arg(-1)+\arg(z) \pmod{2\pi}$$

   Since $\arg(-1) = \pi$ (using the principal argument $(-\pi,\pi]$), we have:

   $$\arg(-z) = \arg(z)+\pi \pmod{2\pi}$$

   This means $\arg(-z)$ is either $\arg(z)+\pi$ or $\arg(z)-\pi$.

   We are given $\arg(z) < 0$. Let $\theta = \arg(z)$, so $\theta \in (-\pi, 0)$.

   Consider the two possibilities for $\arg(-z)$:

   - $\theta+\pi$: Since $\theta \in (-\pi,0)$, we have $\theta+\pi \in (0,\pi)$. This is a valid principal argument.
   - $\theta-\pi$: Since $\theta \in (-\pi,0)$, we have $\theta-\pi \in (-2\pi,-\pi)$. This is outside the principal range $(-\pi,\pi]$.

   Therefore, the principal argument is $\arg(-z) = \arg(z)+\pi$.

   $$\arg(-z)-\arg(z) = \pi$$

   Answer: π

9. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = |\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}| = 1$, then $|z_1 + z_2 + z_3|$ is : A. equal to 1 B. less than 1 C. greater than 3 D. equal to 3

   Solution: Given $|z_1| = |z_2| = |z_3| = 1$.

   For any complex number z with $|z| = 1$, we have $|z|^2 = 1$, which means $z\overline{z} = 1$, so $\overline{z} = \frac{1}{z}$.

   Applying this property to z_1, z_2, z_3:

$$\frac{1}{z_1} = \overline{z_1}, \quad \frac{1}{z_2} = \overline{z_2}, \quad \frac{1}{z_3} = \overline{z_3}$$

   Now consider the given magnitude:

$$\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$$

   Substitute the conjugates:

$$|\overline{z_1} + \overline{z_2} + \overline{z_3}| = 1$$

   We use the property that the magnitude of a sum is equal to the magnitude of its conjugate sum: $|\overline{z_1 + z_2 + z_3}| = |z_1 + z_2 + z_3|$.

$$|\overline{z_1 + z_2 + z_3}| = 1$$
$$|z_1 + z_2 + z_3| = 1$$

   Answer: equal to 1

10. Let z_1 and z_2 be the n^{th} roots of unity which subtend a right angle at the origin, then n must be of the form : A. 4k+1 B. 4k+2 C. 4k+3 D. 4k

   Solution: The n^{th} roots of unity are $z_k = e^{i\frac{2\pi k}{n}}$, where $k = 0, 1, 2, \ldots, n-1$.

   Let z_1 and z_2 be two such roots, corresponding to indices k_1 and k_2.

$$z_1 = e^{i\frac{2\pi k_1}{n}}, \quad z_2 = e^{i\frac{2\pi k_2}{n}}$$

   The angle subtended by z_1 and z_2 at the origin is the difference between their arguments:

$$\arg(z_2) - \arg(z_1) = \frac{2\pi k_2}{n} - \frac{2\pi k_1}{n} = \frac{2\pi}{n}(k_2 - k_1)$$

   For the angle to be a right angle, the difference in arguments must be $\pm\frac{\pi}{2}$.

$$\frac{2\pi}{n}(k_2 - k_1) = \pm\frac{\pi}{2}$$

   Let $k_2 - k_1 = m$, where m is a non-zero integer $(1 \leq |m| \leq n-1)$.

$$\frac{2\pi m}{n} = \frac{\pi}{2}$$
$$\frac{2m}{n} = \frac{1}{2}$$
$$4m = n$$

   Since m must be an integer, n must be a multiple of 4. If n is of the form $4k$ for some integer $k \geq 1$, then $m = k$. For example, if $n = 4k$, we can choose $k_1 = 0$ and $k_2 = k$. The angle is $\frac{2\pi k}{4k} = \frac{\pi}{2}$.

   Thus, n must be of the form $4k$.

   Answer: 4k

11. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1-z_3}{z_2-z_3} = \frac{1-i\sqrt{3}}{2}$ are the vertices of a triangle which is : A. of area zero B. right angled isosceles C. equilateral D. obtuse angled isosceles

Solution: Let $w = \frac{1-i\sqrt{3}}{2}$.

First, find the modulus and argument of w:

$$|w| = \left|\frac{1}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\arg(w) = \arg\left(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})\right) = -\frac{\pi}{3} \quad \text{or} \quad \frac{5\pi}{3}$$

The given equation is $\frac{z_1-z_3}{z_2-z_3} = w$.

1. Modulus:

$$\left|\frac{z_1 - z_3}{z_2 - z_3}\right| = |w| = 1$$
$$|z_1 - z_3| = |z_2 - z_3|$$

This means the distance between z_1 and z_3 is equal to the distance between z_2 and z_3. The side lengths z_3z_1 and z_3z_2 are equal, so the triangle is **isosceles** with z_3 as the vertex angle.

2. Argument:

$$\arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg(w) = -\frac{\pi}{3} \quad \text{or} \quad \frac{5\pi}{3}$$
$$\arg(z_1 - z_3) - \arg(z_2 - z_3) = \angle z_1z_3z_2 = \pm\frac{\pi}{3}$$

The angle at vertex z_3 is $\frac{\pi}{3}$ or $60°$.

Since the triangle is isosceles and the vertex angle is $60°$, the other two angles must also be $60°$: $\frac{180°-60°}{2} = 60°$.

Therefore, the triangle is **equilateral**.

Answer: equilateral

12. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$ is : A. 3ω B. $3\omega(\omega-1)$ C. $3\omega^2$ D. $3\omega(1-\omega)$

Solution: Given $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, ω is a cube root of unity, so $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$.

From $1 + \omega + \omega^2 = 0$, we have $-1 - \omega^2 = \omega$.

Substitute this into the determinant D:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

This is a form of the Vandermonde determinant, but it's simpler to use row/column operations. Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$:

$$D = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \omega - 1 & \omega^2 - 1 \\ 1 & \omega^2 - 1 & \omega - 1 \end{vmatrix}$$

Expand along the first row:

$$\begin{aligned} D &= 1 \cdot \begin{vmatrix} \omega - 1 & \omega^2 - 1 \\ \omega^2 - 1 & \omega - 1 \end{vmatrix} \\ &= (\omega - 1)(\omega - 1) - (\omega^2 - 1)(\omega^2 - 1) \\ &= (\omega - 1)^2 - (\omega^2 - 1)^2 \end{aligned}$$

Using $A^2 - B^2 = (A-B)(A+B)$:

$$\begin{aligned} D &= \big((\omega-1)-(\omega^2-1)\big)\big((\omega-1)+(\omega^2-1)\big) \\ &= (\omega-\omega^2)(\omega+\omega^2-2) \end{aligned}$$

Now use $1+\omega+\omega^2 = 0 \implies \omega+\omega^2 = -1$:

$$\begin{aligned} D &= (\omega-\omega^2)(-1-2) \\ &= (\omega-\omega^2)(-3) \\ &= 3(\omega^2-\omega) = -3(\omega-\omega^2) \end{aligned}$$

Since $3\omega(\omega-1) = 3(\omega^2-\omega)$, the determinant is $3\omega^2 - 3\omega$.

Answer: $3\omega(\omega-1)$

13. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is : A. 0 B. 2 C. 7 D. 17

Solution: $|z_1| = 12$ means z_1 lies on a circle C_1 centered at the origin $O(0)$ with radius $R_1 = 12$.

$|z_2 - 3 - 4i| = 5$ means z_2 lies on a circle C_2 centered at $C(3+4i)$ with radius $R_2 = 5$.

The minimum value of $|z_1 - z_2|$ is the minimum distance between the two circles.

First, find the distance between the centers $O(0)$ and $C(3+4i)$:

$$d = |0-(3+4i)| = |3+4i| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

Now, compare the distance d with the radii R_1 and R_2:

$$R_1 = 12, \quad R_2 = 5, \quad d = 5$$

Since $d < R_1$, the smaller circle C_2 is inside the larger circle C_1. Specifically, $R_1 - R_2 = 12 - 5 = 7$. Since $d = 5 < 7$, C_2 is strictly inside C_1.

The minimum distance between two points on the circles (the shortest distance between the circles) is:

$$d_{\text{min}} = R_1 - (d + R_2)$$

Wait, this formula is only for external distance. The minimum distance is $d_{\text{min}} = R_1 - (d + R_2)$ if one circle is completely contained within the other, but the center of C_2 is not O.

The minimum distance between two circles is:

$$d_{\text{min}} = \text{Distance}(O, C) - R_2 \quad \text{if } C_2 \text{ is outside } C_1$$

$$d_{\text{min}} = R_1 - (d + R_2) \quad \text{if } C_2 \text{ is strictly inside } C_1$$

$$d_{\text{min}} = 0 \quad \text{if the circles intersect or touch}$$

Since $d = 5, R_1 = 12, R_2 = 5$, we have:

$$d + R_2 = 5 + 5 = 10$$

$$R_1 = 12$$

Since $d + R_2 < R_1$ $(10 < 12)$, circle C_2 is entirely contained within C_1.

The minimum distance d_{min} is the distance from the outermost point of C_2 (on the line segment OC) to the circle C_1.

$$d_{\text{min}} = R_1 - (d + R_2) = 12 - (5+5) = 12 - 10 = 2$$

Answer: 2

14. If $|z| = 1$ and w $=\frac{z-1}{z+1}$ (where $z \neq -1$), then Re(w) is : A. 0 B. $\frac{1}{|z+1|^2}$ C. $|\frac{1}{z+1}|.\frac{1}{|z+1|^2}$ D. $\frac{\sqrt{2}}{|z+1|^2}$

Solution: Given $|z| = 1$, we have $z\overline{z} = 1$, so $\frac{1}{z} = \overline{z}$.

To find $\text{Re}(w)$, we use $\text{Re}(w) = \frac{w+\overline{w}}{2}$.

$$\overline{w} = \overline{\left(\frac{z-1}{z+1}\right)} = \frac{\overline{z}-1}{\overline{z}+1}$$

Substitute $\overline{z} = \frac{1}{z}$:

$$\overline{w} = \frac{\frac{1}{z}-1}{\frac{1}{z}+1} = \frac{\frac{1-z}{z}}{\frac{1+z}{z}} = \frac{1-z}{1+z} = -\frac{z-1}{z+1} = -w$$

Since $\overline{w} = -w$, we have $w + \overline{w} = 0$.

$$\text{Re}(w) = \frac{w+\overline{w}}{2} = \frac{0}{2} = 0$$

This means w is purely imaginary. This is a known property: the image of the unit circle $|z| = 1$ (excluding $z = -1$) under the Mobius transformation $w = \frac{z-1}{z+1}$ is the imaginary axis $\text{Re}(w) = 0$.

Answer: 0

15. If $\omega(\neq 1)$ be a cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, then the least positive value of n is A. 2 B. 3 C. 5 D. 6

Solution: Since ω is a cube root of unity, $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$.

Simplify the bases:

- $1 + \omega^2 = -\omega \quad (\text{from } 1 + \omega + \omega^2 = 0)$
- $1 + \omega^4 = 1 + \omega^3 \cdot \omega = 1 + \omega \quad (\text{from } \omega^3 = 1)$
- $1 + \omega = -\omega^2 \quad (\text{from } 1 + \omega + \omega^2 = 0)$

The equation becomes:

$$(-\omega)^n = (-\omega^2)^n$$
$$(-1)^n\omega^n = (-1)^n(\omega^2)^n$$

If n is even, $(-1)^n = 1$, and the equation is $\omega^n = \omega^{2n}$.

$$\omega^{2n} - \omega^n = 0$$

$$\omega^n(\omega^n - 1) = 0$$

Since $\omega \neq 0$, we must have $\omega^n = 1$. The smallest positive integer n for which $\omega^n = 1$ is $n = 3$.

If n is odd, $(-1)^n = -1$, and the equation is $-\omega^n = -\omega^{2n}$, which also leads to $\omega^n = \omega^{2n}$ and thus $\omega^n = 1$.

The least positive value of n for which $\omega^n = 1$ is $n = 3$.

Answer: 3

16. The minimum value of $|a + b\omega + c\omega^2|$, where a,b and c are all not equal integers and $\omega(\neq 1)$ is a cube root of unity, is : A. $\sqrt{3}$ B. $\frac{1}{2}$ C. 1 D. 0

Solution: Let $z = a + b\omega + c\omega^2$. We want to find $|z|^2$ and then minimize it.

Since $1 + \omega + \omega^2 = 0$, we have $\omega^2 = -1 - \omega$.

$$z = a + b\omega + c(-1-\omega) = a - c + (b - c)\omega$$

Substitute $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$:

$$\begin{aligned} z &= (a-c) + (b-c)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= \left(a - c - \frac{b-c}{2}\right) + i\left(\frac{(b-c)\sqrt{3}}{2}\right) \\ &= \left(\frac{2a-2c-b+c}{2}\right) + i\left(\frac{(b-c)\sqrt{3}}{2}\right) \\ &= \frac{2a-b-c}{2} + i\frac{\sqrt{3}(b-c)}{2} \end{aligned}$$

Now compute $|z|^2$:

$$\begin{aligned} |z|^2 &= \left(\frac{2a-b-c}{2}\right)^2 + \left(\frac{\sqrt{3}(b-c)}{2}\right)^2 \\ &= \frac{1}{4}\left((2a-b-c)^2 + 3(b-c)^2\right) \end{aligned}$$

Since a, b, c are integers that are not all equal, $2a-b-c$ and $b-c$ must be integers.

For $|z|^2$ to be minimum, the expression $(2a-b-c)^2 + 3(b-c)^2$ must be minimum and non-zero (since a, b, c are not all equal, z cannot be zero).

The minimum value of a sum of squares of integers is achieved when the integers are small. Since a, b, c are not all equal, $b - c \neq 0$ or $2a - b - c \neq 0$.

1. If $b - c = 0$, then $b = c$. Since a, b, c are not all equal, $a \neq b$.

$$|z|^2 = \frac{1}{4}(2a-b-b)^2 + 0 = \frac{1}{4}(2a-2b)^2 = \frac{1}{4}(2(a-b))^2 = \frac{1}{4}\cdot 4(a-b)^2 = (a-b)^2$$

Since $a - b$ is a non-zero integer, the smallest possible value for $(a-b)^2$ is $1^2 = 1$. In this case, $|z|^2 = 1$, and $|z| = 1$. (e.g., $a = 1, b = 0, c = 0$: $|1 + 0\omega + 0\omega^2| = 1$)

2. If $b - c \neq 0$, the smallest possible value for $(b-c)^2$ is 1. Let $b - c = \pm 1$.

If we take $b - c = 1$:

$$|z|^2 = \frac{1}{4}((2a-b-c)^2 + 3(1))$$

We want to minimize the integer $(2a-b-c)^2$. Since $b - c = 1$, $b = c + 1$.

$$2a - b - c = 2a - (c+1) - c = 2a - 2c - 1 = 2(a-c) - 1$$

Since $a - c$ is an integer, $2(a-c)$ is an even integer, and $2(a-c) - 1$ is an odd integer. The smallest non-zero value for an odd integer squared is $(\pm 1)^2 = 1$. We can achieve $2(a-c) - 1 = 1$ by taking $a - c = 1$. (e.g., $a = 2, c = 1, b = 2$).

If $a - c = 1$, then:

$$|z|^2 = \frac{1}{4}(1^2 + 3(1)^2) = \frac{1}{4}(1+3) = 1$$

In both cases, the minimum value of $|z|^2$ is 1.

The minimum value of $|a + b\omega + c\omega^2|$ is $\sqrt{1} = 1$.

Answer: 1