Binomial Theorem - Set 2: DETAILED SOLUTIONS

1. r and n are positive integers $r > 1, n > 2$ and coefficients of $(r+2)^{\text{th}}$ term and $3r^{\text{th}}$ term in the expansion of $(1+x)^{2n}$ are equal, then n equals

   Solution: The general term in $(1+x)^N$ is T_{k+1} with coefficient C_k^N. Here $N = 2n$.

   - Coefficient of $(r+2)^{\text{th}}$ term ($k_1 = r+1$): C_{r+1}^{2n}
   - Coefficient of $3r^{\text{th}}$ term ($k_2 = 3r-1$): C_{3r-1}^{2n}

   Given: $C_{r+1}^{2n} = C_{3r-1}^{2n}$.
   The equality $C_a^N = C_b^N$ holds if $a = b$ or $a + b = N$.

   (a) Case 1: $r + 1 = 3r - 1 \implies 2 = 2r \implies r = 1$. This contradicts the condition $r > 1$, so we reject this case.
   (b) Case 2: $(r+1) + (3r-1) = 2n$
   $$4r = 2n$$
   $$n = 2r$$

   (a) 3r
   (b) 3r +1
   (c) **2r**
   (d) 2r +1

   [Ans. c]

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2. If x is positive, the first negative term in the expansion of $(1+x)^{\frac{27}{5}}$ is

   Solution: This is a binomial expansion for a fractional index ($\alpha = \frac{27}{5} = 5.4$). The general term is $T_{r+1} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}x^r$.

   For T_{r+1} to be the first negative term, the coefficient must be negative, meaning the numerator $\alpha(\alpha-1)\cdots(\alpha-r+1)$ must be negative.

   Since $\alpha > 0, \alpha - 1 > 0, \ldots$, the product will become negative when the first negative factor appears. The last factor is $(\alpha - r + 1)$. We require:

   $$\alpha - r + 1 < 0$$
   $$\frac{27}{5} - r + 1 < 0$$
   $$5.4 - r + 1 < 0$$
   $$6.4 - r < 0$$
   $$r > 6.4$$

   Since r must be an integer, the smallest integer value for r is $r = 7$. The first negative term is $T_{r+1} = T_{7+1} = T_8$, the **8th term**.

   (a) 6th term
   (b) 7th term
   (c) 5th term
   (d) **8th term**

   [Ans. d]

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3. The coefficient of the middle term in the binomial expansion in powers of x of $(1+\alpha x)^4$ and $(1-x\alpha)^6$ is the same is α equals

Solution: 1. Expansion of $(1+\alpha x)^4$**:** ($n=4$, even). The middle term is $T_{\frac{4}{2}+1} = T_3$.

$$T_3 = C_2^4(1)^{4-2}(\alpha x)^2 = 6\alpha^2 x^2$$

Coefficient of the middle term: $C_{M1} = 6\alpha^2$.

2. Expansion of $(1-\alpha x)^6$**:** ($n=6$, even). The middle term is $T_{\frac{6}{2}+1} = T_4$.

$$T_4 = C_3^6(1)^{6-3}(-\alpha x)^3 = 20(-\alpha^3 x^3) = -20\alpha^3 x^3$$

Coefficient of the middle term: $C_{M2} = -20\alpha^3$.

Given $C_{M1} = C_{M2}$:

$$6\alpha^2 = -20\alpha^3$$
$$20\alpha^3 + 6\alpha^2 = 0$$
$$2\alpha^2(10\alpha + 3) = 0$$

Possible solutions are $\alpha = 0$ or $10\alpha + 3 = 0$. If $\alpha = 0$, the expansion is trivial. Assuming $\alpha \neq 0$:

$$10\alpha = -3 \implies \alpha = -\frac{3}{10}$$

(a) $\frac{3}{5}$
(b) $\frac{10}{3}$
(c) $\mathbf{\frac{-3}{10}}$
(d) $\frac{-5}{3}$

[Ans. c]

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4. The coefficient of x^n in expansion of $(1+x)(1-x)^n$ is

Solution:

$$(1+x)(1-x)^n = 1\cdot(1-x)^n + x\cdot(1-x)^n$$

We need the coefficient of x^n in the whole expression.

1. Coefficient of x^n **in** $1\cdot(1-x)^n$**:** The general term of $(1-x)^n$ is $C_r^n(-x)^r = C_r^n(-1)^r x^r$. For x^n, we set $r=n$.

$$\text{Coeff}_1 = C_n^n(-1)^n = 1\cdot(-1)^n = (-1)^n$$

2. Coefficient of x^n **in** $x\cdot(1-x)^n$**:**

$$x\cdot(1-x)^n = x\cdot\left[\sum_{r=0}^{n} C_r^n(-1)^r x^r\right] = \sum_{r=0}^{n} C_r^n(-1)^r x^{r+1}$$

For the term to contain x^n, we need $r+1=n$, so $r=n-1$.

$$\text{Coeff}_2 = C_{n-1}^n(-1)^{n-1}$$

Since $C_{n-1}^n = C_1^n = n$:

$$\text{Coeff}_2 = n(-1)^{n-1}$$

3. Total Coefficient of x^n**:**

$$\text{Total Coeff} = \text{Coeff}_1 + \text{Coeff}_2 = (-1)^n + n(-1)^{n-1}$$

Factor out $(-1)^{n-1}$:

$$\text{Total Coeff} = (-1)^{n-1}\left[(-1)+n\right] = (-1)^{n-1}(n-1)$$

Alternatively, factor out $(-1)^n$:

$$\text{Total Coeff} = (-1)^n\left[1 + n(-1)^{-1}\right] = (-1)^n\left[1-n\right]$$

(a) $(-1)^{n-1}n$
(b) $\mathbf{(-1)^n(1-n)}$
(c) $(-1)^{n-1}(n-1)^2$
(d) $(n-1)$

[Ans. b]
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5. The value of $^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is
Solution: Let $E = C_4^{50} + \sum_{r=1}^{6} C_3^{56-r}$.
Expand the sum:

$$\sum_{r=1}^{6} C_3^{56-r} = C_3^{55} + C_3^{54} + C_3^{53} + C_3^{52} + C_3^{51} + C_3^{50}$$

The expression E is:

$$E = C_4^{50} + C_3^{50} + C_3^{51} + C_3^{52} + C_3^{53} + C_3^{54} + C_3^{55}$$

Apply Pascal's Identity $C_r^n + C_{r-1}^n = C_r^{n+1}$ repeatedly, starting with the first two terms:

$$\begin{aligned} E &= (C_4^{50} + C_3^{50}) + C_3^{51} + C_3^{52} + C_3^{53} + C_3^{54} + C_3^{55} \\ &= C_4^{51} + C_3^{51} + C_3^{52} + C_3^{53} + C_3^{54} + C_3^{55} \\ &= (C_4^{51} + C_3^{51}) + C_3^{52} + C_3^{53} + C_3^{54} + C_3^{55} \\ &= C_4^{52} + C_3^{52} + C_3^{53} + C_3^{54} + C_3^{55} \\ &= C_4^{53} + C_3^{53} + C_3^{54} + C_3^{55} \\ &= C_4^{54} + C_3^{54} + C_3^{55} \\ &= C_4^{55} + C_3^{55} \\ &= C_4^{56} \end{aligned}$$

(a) $^{55}C_4$
(b) $^{55}C_3$
(c) $^{56}C_3$
(d) $\mathbf{^{56}C_4}$

[Ans. d]
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6. The sum of the series $^{20}C_0 -^{20} C_1 +^{20} C_2 -^{20} C_3 + \cdots - \cdots +^{20} C_{10}$ is
Solution: The complete alternating sum is:

$$S_{\text{full}} = C_0^{20} - C_1^{20} + C_2^{20} - \cdots + C_{20}^{20} = (1-1)^{20} = 0$$

The required sum is $S = C_0^{20} - C_1^{20} + \cdots + C_{10}^{20}$.
We use the property $C_r^n = C_{n-r}^n$.

$$S_{\text{full}} = (C_0^{20} - C_1^{20} + \cdots - C_9^{20} + C_{10}^{20}) + (-C_{11}^{20} + C_{12}^{20} - \cdots + C_{20}^{20}) = 0$$

Consider the terms from C_{11}^{20} onwards:

$$\begin{aligned} C_{11}^{20} &= C_9^{20} \\ C_{12}^{20} &= C_8^{20} \\ &\vdots \\ C_{20}^{20} &= C_0^{20} \end{aligned}$$

The sum can be written as:

$$\begin{aligned} S_{\text{full}} &= (C_0^{20} - C_1^{20} + \cdots - C_9^{20} + C_{10}^{20}) + (-C_9^{20} + C_8^{20} - \cdots + C_0^{20}) \\ &= (C_0^{20} - C_1^{20} + \cdots - C_9^{20}) + C_{10}^{20} + (C_0^{20} - C_1^{20} + \cdots - C_9^{20}) \\ &= 2(C_0^{20} - C_1^{20} + \cdots - C_9^{20}) + C_{10}^{20} = 0 \end{aligned}$$

Let $S_A = C_0^{20} - C_1^{20} + \cdots - C_9^{20}$. Then $2S_A + C_{10}^{20} = 0 \implies S_A = -\frac{1}{2}C_{10}^{20}$.
The required sum S is:

$$S = S_A + C_{10}^{20} = -\frac{1}{2}C_{10}^{20} + C_{10}^{20} = \frac{1}{2}C_{10}^{20}$$

(a) 0
(b) ${}^{20}C_{10}$
(c) $-{}^{20}C_{10}$
(d) $\mathbf{\frac{1}{2}{}^{20}C_{10}}$

[Ans. d]

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7. If n is a positive integer, then $(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}$ is

Solution: Let $X = (\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}$. Let $A = \sqrt{3}$ and $B = 1$. Using the binomial expansion:

$$(A+B)^N - (A-B)^N = 2 \sum_{\substack{r=1 \\ r \text{ odd}}}^{N} C_r^N A^{N-r} B^r$$

Here $N = 2n$ (even).

$$X = 2\left[C_1^{2n}(\sqrt{3})^{2n-1}(1)^1 + C_3^{2n}(\sqrt{3})^{2n-3}(1)^3 + \cdots + C_{2n-1}^{2n}(\sqrt{3})^1(1)^{2n-1}\right]$$

The term inside the bracket is:

$$C_1^{2n} 3^{\frac{2n-1}{2}} + C_3^{2n} 3^{\frac{2n-3}{2}} + \cdots + C_{2n-1}^{2n} 3^{1/2}$$

Since the exponents of 3, $\frac{2n-1}{2}, \frac{2n-3}{2}, \ldots, \frac{1}{2}$, are all half-integers, the powers of $\sqrt{3}$ are odd powers of $\sqrt{3}$, e.g., $3^{k-1/2}\sqrt{3}$.
Let I be the sum in the bracket. Each term contains a factor of $\sqrt{3}$:

$$I = \sqrt{3}\left[C_1^{2n}3^{n-1} + C_3^{2n}3^{n-2} + \cdots + C_{2n-1}^{2n}3^0\right]$$

The expression in the square brackets is a sum of integers, hence it is an integer. Let this integer be K.

$$X = 2\sqrt{3}K$$

Since n is a positive integer, K is a non-zero integer.
The expression X is $2 \times$ integer $\times \sqrt{3}$, which is a product of a non-zero integer and the irrational number $\sqrt{3}$. Therefore, X is an **irrational number**.

(a) **an irrational number**
(b) an odd positive integer
(c) an even positive integer
(d) a rational number other than positive integers

[Ans. a]

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8. Expand by binomial theorem: $(1-x+x^2)^4$

Solution: We group the terms and use the binomial theorem, $\big((1-x)+x^2\big)^4$:

$$\big((1-x)+x^2\big)^4 = \sum_{r=0}^{4} C_r^4 (1-x)^{4-r}(x^2)^r$$

$$\begin{aligned}
T_1(r=0) &: C_0^4(1-x)^4(x^2)^0 = 1\cdot(1-x)^4 \\
T_2(r=1) &: C_1^4(1-x)^3(x^2)^1 = 4x^2(1-x)^3 \\
T_3(r=2) &: C_2^4(1-x)^2(x^2)^2 = 6x^4(1-x)^2 \\
T_4(r=3) &: C_3^4(1-x)^1(x^2)^3 = 4x^6(1-x) \\
T_5(r=4) &: C_4^4(1-x)^0(x^2)^4 = 1\cdot x^8
\end{aligned}$$

Expanding each term:

$$\begin{aligned}
(1-x)^4 &= 1-4x+6x^2-4x^3+x^4 \\
4x^2(1-x)^3 &= 4x^2(1-3x+3x^2-x^3) = 4x^2-12x^3+12x^4-4x^5 \\
6x^4(1-x)^2 &= 6x^4(1-2x+x^2) = 6x^4-12x^5+6x^6 \\
4x^6(1-x) &= 4x^6-4x^7 \\
x^8 &= x^8
\end{aligned}$$

Adding the terms by powers of x:

$$\begin{aligned}
x^0 &: \ 1 \\
x^1 &: \ -4x \\
x^2 &: \ 6x^2+4x^2 = 10x^2 \\
x^3 &: \ -4x^3-12x^3 = -16x^3 \\
x^4 &: \ x^4+12x^4+6x^4 = 19x^4 \\
x^5 &: \ -4x^5-12x^5 = -16x^5 \\
x^6 &: \ 6x^6+4x^6 = 10x^6 \\
x^7 &: \ -4x^7 \\
x^8 &: \ x^8
\end{aligned}$$

$$(1-x+x^2)^4 = 1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$$

[Ans. $1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$]

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9. Find the term independent of x in $(x^2+\frac{1}{x})^9$

Solution: The general term is $T_{r+1} = C_r^n a^{n-r} b^r$. Here $n=9$, $a=x^2$, $b=x^{-1}$.

$$T_{r+1} = C_r^9 (x^2)^{9-r}(x^{-1})^r$$

$$T_{r+1} = C_r^9 x^{18-2r} x^{-r} = C_r^9 x^{18-3r}$$

For the term to be independent of x, the exponent of x must be zero:

$$18-3r=0 \implies 3r=18 \implies r=6$$

The term independent of x is $T_{r+1} = T_{6+1} = T_7$, the **7th term**.

[Ans. 7th term]

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10. If coefficients of a^{r-1}, a^r, a^{r+1} in $(1+a)^n$ are in A.P., prove:

$$n^2 - n(4r+1) + 4r^2 - 2 = 0$$

Proof: The coefficient of a^k in $(1+a)^n$ is C_k^n. The coefficients of a^{r-1}, a^r, a^{r+1} are:

$$C_{r-1} = C_{r-1}^n, \quad C_r = C_r^n, \quad C_{r+1} = C_{r+1}^n$$

If they are in A.P., then $2C_r = C_{r-1} + C_{r+1}$.

$$2C_r^n = C_{r-1}^n + C_{r+1}^n$$

Divide the equation by C_{r-1}^n:

$$2\frac{C_r^n}{C_{r-1}^n} = 1 + \frac{C_{r+1}^n}{C_{r-1}^n}$$

Using the ratio identities $\frac{C_r^n}{C_{r-1}^n} = \frac{n-r+1}{r}$ and $\frac{C_{r+1}^n}{C_{r-1}^n} = \frac{C_{r+1}^n}{C_r^n} \cdot \frac{C_r^n}{C_{r-1}^n} = \frac{n-(r+1)+1}{r+1} \cdot \frac{n-r+1}{r} = \frac{n-r}{r+1} \cdot \frac{n-r+1}{r}$:

$$2\left(\frac{n-r+1}{r}\right) = 1 + \left(\frac{n-r}{r+1}\right)\left(\frac{n-r+1}{r}\right)$$

Multiply by $r(r+1)$ to clear denominators:

$$2(r+1)(n-r+1) = r(r+1) + (n-r)(n-r+1)$$

Expand the terms:

$$2(nr - r^2 + r + n - r + 1) = r^2 + r + (n-r)(n-r+1)$$

$$2(nr - r^2 + n + 1) = r^2 + r + [n^2 - nr + n - nr + r^2 - r]$$

$$2nr - 2r^2 + 2n + 2 = r^2 + r + n^2 - 2nr + n + r^2 - r$$

Group all terms on the right side:

$$0 = n^2 + (2nr - 2nr) + r^2 + r^2 + n - 2n + r - r - 2nr + 2r^2 - 2$$

$$0 = n^2 + (21n - 22n) + (2r^2 - 2r^2)$$

Re-evaluation (Simplifying powers of r):

$$2nr - 2r^2 + 2n + 2 = n^2 - 2nr + 2r^2 + n$$

Move all terms to the right side:

$$0 = n^2 - 2nr - 2nr + 2r^2 + 2r^2 + n - 2n - 2$$

$$0 = n^2 - 4nr + 4r^2 - n - 2$$

Rearrange to match the required form:

$$n^2 - n(4r+1) + 4r^2 - 2 = 0 \quad \text{(Proved)}$$

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11. If in $(1+x)^m(1-x)^n$, coeff. of x is 3 and of x^2 is -6, find m.

Solution: We consider the first few terms of the expansions:

$$(1+x)^m = 1 + C_1^m x + C_2^m x^2 + \cdots = 1 + mx + \frac{m(m-1)}{2}x^2 + \cdots$$

$$(1-x)^n = 1 - C_1^n x + C_2^n x^2 - \cdots = 1 - nx + \frac{n(n-1)}{2}x^2 - \cdots$$

The product $P(x) = (1+x)^m(1-x)^n$:

$$P(x) = \left(1 + mx + \frac{m(m-1)}{2}x^2 + \cdots\right)\left(1 - nx + \frac{n(n-1)}{2}x^2 - \cdots\right)$$

1. Coefficient of x: Coeff of x comes from $1 \cdot (-nx)$ and $(mx) \cdot 1$:

$$\text{Coeff}(x) = -n + m$$

Given $\text{Coeff}(x) = 3$:

$$m - n = 3 \quad (\mathbf{1})$$

2. Coefficient of x^2: Coeff of x^2 comes from $1 \cdot \left(\frac{n(n-1)}{2}x^2\right)$, $(mx) \cdot (-nx)$, and $\left(\frac{m(m-1)}{2}x^2\right) \cdot 1$:

$$\text{Coeff}(x^2) = \frac{n(n-1)}{2} - mn + \frac{m(m-1)}{2}$$

Given $\text{Coeff}(x^2) = -6$:

$$\frac{n^2-n}{2} - mn + \frac{m^2-m}{2} = -6$$

Multiply by 2:

$$n^2 - n - 2mn + m^2 - m = -12$$
$$(m^2 - 2mn + n^2) - (m+n) = -12$$
$$(m-n)^2 - (m+n) = -12 \quad (\mathbf{2})$$

Substitute (1) into (2):

$$(3)^2 - (m+n) = -12$$
$$9 - (m+n) = -12$$
$$m + n = 9 + 12 = 21 \quad (\mathbf{3})$$

Solve the system of equations (1) and (3) for m:

$$m - n = 3$$
$$m + n = 21$$

Adding the two equations:

$$2m = 24 \implies m = 12$$

(Then $n = 21 - 12 = 9$).

[Ans. 12]

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12. Find ${}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \cdots + {}^{4n}C_{4n}$

Solution: Let $S = C_0^{4n} + C_4^{4n} + C_8^{4n} + \cdots + C_{4n}^{4n}$. This is the sum of coefficients C_r^{4n} where $r \equiv 0 \pmod 4$.

We use the root of unity method with $\omega = i$ and $\omega = -i$, which are the 4th roots of unity (excluding ± 1). Let $(1+x)^{4n} = \sum_{r=0}^{4n} C_r^{4n} x^r$.

Set $x = 1$: $C_0^{4n} + C_1^{4n} + C_2^{4n} + C_3^{4n} + \cdots + C_{4n}^{4n} = 2^{4n} \quad (\mathbf{1})$ Set $x = -1$: $C_0^{4n} - C_1^{4n} + C_2^{4n} - C_3^{4n} + \cdots + C_{4n}^{4n} = 0 \quad (\mathbf{2})$

Set $x = i$:

$$(1+i)^{4n} = C_0^{4n} + C_1^{4n} i - C_2^{4n} - C_3^{4n} i + C_4^{4n} + \cdots \quad \mathbf{(3)}$$

Set $x = -i$:

$$(1-i)^{4n} = C_0^{4n} - C_1^{4n} i - C_2^{4n} + C_3^{4n} i + C_4^{4n} + \cdots \quad \mathbf{(4)}$$

Adding (3) and (4):

$$(1+i)^{4n} + (1-i)^{4n} = 2[C_0^{4n} - C_2^{4n} + C_4^{4n} - C_6^{4n} + \cdots] \quad \mathbf{(5)}$$

Calculate the L.H.S. of (5):

$$(1+i) = \sqrt{2}e^{i\pi/4} \implies (1+i)^{4n} = (\sqrt{2})^{4n} e^{in\pi} = 2^{2n}(\cos(n\pi) + i\sin(n\pi)) = 2^{2n}(-1)^n$$
$$(1-i) = \sqrt{2}e^{-i\pi/4} \implies (1-i)^{4n} = 2^{2n}e^{-in\pi} = 2^{2n}(-1)^n$$
$$(1+i)^{4n} + (1-i)^{4n} = 2 \cdot 2^{2n}(-1)^n = 2^{2n+1}(-1)^n$$

From (5): $C_0^{4n} - C_2^{4n} + C_4^{4n} - C_6^{4n} + \cdots = \frac{1}{2}(2^{2n+1}(-1)^n) = 2^{2n}(-1)^n \quad \mathbf{(6)}$

Add (1), (2), and twice of (6):

$$(1) + (2) + 2 \times (6):$$

The sum of all coefficients + alternating sum of all coefficients + twice the even-indexed alternating sum.

$$[C_0 + C_1 + C_2 + \cdots] + [C_0 - C_1 + C_2 - \cdots] + 2[C_0 - C_2 + C_4 - \cdots]$$
$$4C_0^{4n} + 4C_4^{4n} + 4C_8^{4n} + \cdots = 2^{4n} + 0 + 2 \cdot 2^{2n}(-1)^n$$

$$4S = 2^{4n} + 2^{2n+1}(-1)^n$$
$$S = \frac{2^{4n}}{4} + \frac{2^{2n+1}}{4}(-1)^n$$
$$S = 2^{4n-2} + 2^{2n-1}(-1)^n$$

[Ans. $2^{4n-2} + (-1)^n 2^{2n-1}$]

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13. If $a_k = \frac{1}{k(k+1)}$, prove $\left(\sum_{k=1}^{n} a_k\right)^2 = \left(\frac{n}{n+1}\right)^2$

Proof: First, find the sum $\sum_{k=1}^{n} a_k$. Use partial fraction decomposition for a_k:

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

The sum is a telescoping series:

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$
$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

All intermediate terms cancel out:

$$\sum_{k=1}^{n} a_k = \frac{1}{1} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

Now, square the result:

$$\left(\sum_{k=1}^{n} a_k\right)^2 = \left(\frac{n}{n+1}\right)^2$$

$$\text{LHS} = \text{RHS} \quad \text{(Proved)}$$

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14. If $(r+1)$th term of $\left(\frac{a^{1/3}}{b^{1/6}} + \frac{b^{1/2}}{a^{1/6}}\right)^{21}$ has equal powers of a, b, find r.

Solution: The expansion is $(A+B)^n$ where $n = 21$, $A = \frac{a^{1/3}}{b^{1/6}} = a^{1/3}b^{-1/6}$, and $B = \frac{b^{1/2}}{a^{1/6}} = a^{-1/6}b^{1/2}$.

The general term is $T_{r+1} = C_r^{21} A^{21-r} B^r$:

$$T_{r+1} = C_r^{21} (a^{1/3}b^{-1/6})^{21-r} (a^{-1/6}b^{1/2})^r$$

Collect the powers of a and b:

$$T_{r+1} = C_r^{21} a^{\frac{1}{3}(21-r) - \frac{1}{6}r} b^{-\frac{1}{6}(21-r) + \frac{1}{2}r}$$

Power of a (P_a):

$$P_a = \frac{21-r}{3} - \frac{r}{6} = \frac{2(21-r)-r}{6} = \frac{42-2r-r}{6} = \frac{42-3r}{6}$$

Power of b (P_b):

$$P_b = -\frac{21-r}{6} + \frac{r}{2} = \frac{-(21-r)+3r}{6} = \frac{-21+r+3r}{6} = \frac{4r-21}{6}$$

Given that the powers of a and b are equal: $P_a = P_b$.

$$\frac{42-3r}{6} = \frac{4r-21}{6}$$
$$42 - 3r = 4r - 21$$
$$42 + 21 = 4r + 3r$$
$$63 = 7r \implies r = 9$$

[Ans. 9]

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15. Coefficient of x^{20} in $(1+3x+3x^2+x^3)^{20}$

Solution: Recognize the term inside the bracket as a binomial expansion:

$$1 + 3x + 3x^2 + x^3 = C_0^3(1)^3 + C_1^3(1)^2x + C_2^3(1)x^2 + C_3^3x^3 = (1+x)^3$$

The given expression is:

$$E = \left((1+x)^3\right)^{20} = (1+x)^{60}$$

The coefficient of x^{20} in the expansion of $(1+x)^{60}$ is C_{20}^{60}:

$$\text{Coeff}(x^{20}) = C_{20}^{60} = \frac{60!}{20!40!}$$

[Ans. ${}^{60}C_{40}$]

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16. Two consecutive equal coefficients in $(3+2x)^{74}$

Solution: The general term of $(a+b)^n$ is $T_{r+1} = C_r^n a^{n-r} b^r$. Here $n = 74$, $a = 3$, $b = 2x$. The coefficients are $C_r = C_r^{74} 3^{74-r} 2^r$.

Let the two consecutive equal coefficients be C_r and C_{r+1}.

$$C_r = C_{r+1}$$

$$C_r^{74} 3^{74-r} 2^r = C_{r+1}^{74} 3^{74-(r+1)} 2^{r+1}$$

Expand the terms and simplify:

$$\frac{C_r^{74}}{C_{r+1}^{74}} \cdot \frac{3^{74-r}}{3^{73-r}} \cdot \frac{2^r}{2^{r+1}} = 1$$

Using the identity $\frac{C_r^n}{C_{r+1}^n} = \frac{r+1}{n-r}$:

$$\frac{r+1}{74-r} \cdot 3^1 \cdot 2^{-1} = 1$$

$$\frac{r+1}{74-r} \cdot \frac{3}{2} = 1$$

$$3(r+1) = 2(74-r)$$

$$3r + 3 = 148 - 2r$$

$$5r = 145$$

$$r = \frac{145}{5} = 29$$

The two consecutive equal coefficients correspond to the terms T_{r+1} and T_{r+2}. The value $r = 29$ means the equal coefficients are C_{29} and C_{30}. These are the coefficients of the $(29+1) = 30^{\text{th}}$ term and the $(29+2) = 31^{\text{st}}$ term.

[Ans. 30, 31]

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17. Coefficient of x^r in $(1-x)^{-2}$

Solution: The expansion of $(1-x)^{-n}$ using the Generalized Binomial Theorem is:

$$(1-x)^{-n} = \sum_{r=0}^{\infty} C_r^{n+r-1} x^r$$

For $n = 2$:

$$(1-x)^{-2} = \sum_{r=0}^{\infty} C_r^{2+r-1} x^r = \sum_{r=0}^{\infty} C_r^{r+1} x^r$$

Since $C_r^{r+1} = C_{(r+1)-r}^{r+1} = C_1^{r+1} = r+1$.

$$(1-x)^{-2} = \sum_{r=0}^{\infty} (r+1)x^r = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

The coefficient of x^r is $r+1$.

[Ans. $r+1$]