# Class 12 Mathematics — Detailed Answers & Solutions (LaTeX)

Each part: *Answer* first, then *Detailed solution*, and a short *Verification* that the solution matches the answer.

## Question 1 ($10 \times 1 = 10$ marks)

1. **Answer:** $\boxed{3}$.

   **Solution:** The relation $R$ on $\mathbb{Z}$ is defined by $aRb$ iff $a - b$ is divisible by 3. This is the congruence modulo 3. The equivalence classes are

   $$\overline{0} = \{\, x \in \mathbb{Z} : x \equiv 0 \pmod 3 \,\}, \qquad \overline{1} = \{\, x \in \mathbb{Z} : x \equiv 1 \pmod 3 \,\}, \qquad \overline{2} = \{\, x \in \mathbb{Z} : x \equiv 2 \pmod 3$$

   These three classes are distinct and every integer belongs to exactly one of them. Hence there are 3 distinct equivalence classes.

   **Verification:** The classes $\overline{0}, \overline{1}, \overline{2}$ partition $\mathbb{Z}$ and are distinct, so the count 3 is correct.

2. **Answer:** $\boxed{625}$.

   **Solution:** Let $A$ be a $3 \times 3$ matrix with $|A| = -5$. For an $n \times n$ matrix $M$,

   $$|\operatorname{adj} M| = |M|^{n-1}.$$

   Here $n = 3$. For $A^2$ we have $|A^2| = |A|^2 = (-5)^2 = 25$. Therefore

   $$|\operatorname{adj}(A^2)| = |A^2|^{n-1} = 25^{\,2} = 625.$$

   **Verification:** Computation: $|A| = -5 \Rightarrow |A^2| = 25$. Then $25^2 = 625$, matching the answer.

3. **Answer:** $\boxed{\frac{3}{4}}$.

   **Solution:** Let $\theta = \tan^{-1}\left(\frac{1}{3}\right)$. Then $\tan\theta = \frac{1}{3}$. Use the double-angle formula

   $$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2 \cdot \frac{1}{3}}{1 - \frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2}{3} \cdot \frac{9}{8} = \frac{3}{4}.$$

   **Verification:** Direct substitution yields $\frac{3}{4}$, which equals the answer.

4. **Answer:** $\boxed{k = \pm 1}$.

**Solution:** We require continuity of

$$f(x) = \begin{cases} \dfrac{1-\cos(kx)}{x^2}, & x \neq 0, \\ \dfrac{1}{2}, & x = 0, \end{cases}$$

at $x = 0$. Evaluate the limit $L = \lim_{x\to 0} \dfrac{1-\cos(kx)}{x^2}$. Using the Taylor expansion $\cos t = 1 - \frac{t^2}{2} + o(t^2)$ as $t \to 0$, with $t = kx$,

$$1 - \cos(kx) = \frac{k^2x^2}{2} + o(x^2) \quad \Rightarrow \quad \frac{1-\cos(kx)}{x^2} \to \frac{k^2}{2}.$$

For continuity at 0 we need $L = \frac{1}{2}$. Hence

$$\frac{k^2}{2} = \frac{1}{2} \quad \Rightarrow \quad k^2 = 1 \quad \Rightarrow \quad k = \pm 1.$$

**Verification:** For $k = \pm 1$ the limit equals $1/2$, matching $f(0)$. For other $k$ it does not; thus $k = \pm 1$ is correct.

5. **Answer:** $\boxed{\text{Order } = 3, \text{ Degree } = 4}$.

**Solution:** The differential equation is

$$\left(\frac{d^3y}{dx^3}\right)^2 - 5\frac{d^2y}{dx^2} + \sqrt{\frac{dy}{dx} + 1} = 0.$$

*Order:* the highest derivative appearing is $\dfrac{d^3y}{dx^3}$, so the order is 3.

*Degree:* degree is defined as the power of the highest order derivative after the equation is made a polynomial equation in derivatives (i.e. after removing radicals and fractional powers). The given equation contains $(\dfrac{d^3y}{dx^3})^2$ and also a square root in a lower-order term. To remove the radical we can isolate and square, e.g.

$$\left(\frac{d^3y}{dx^3}\right)^2 - 5\frac{d^2y}{dx^2} = -\sqrt{\frac{dy}{dx} + 1}.$$

Squaring both sides yields a polynomial equation in the derivatives whose highest power of $\dfrac{d^3y}{dx^3}$ is $(\dfrac{d^3y}{dx^3})^4$. Thus the degree (after clearing radicals) is 4.

**Verification:** Order 3 is immediate; after eliminating the radical, the highest power of $y^{(3)}$ becomes 4, so degree 4 is correct.

6. **Answer:** $\boxed{\dfrac{\pi}{4}}$.

   **Solution:** Standard integral:

$$\int_0^1 \frac{dx}{1+x^2} = \big[\tan^{-1} x\big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

   **Verification:** The antiderivative is $\tan^{-1} x$, giving $\pi/4$.

7. **Answer:** $\boxed{\text{Not one-one; Not onto (as a map } \mathbb{R} \to \mathbb{R}).}$

   **Solution:** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x-1|$.

   *Not one-one:* If a function is one-one then distinct inputs give distinct outputs. But

$$f(0) = |0-1| = 1, \qquad f(2) = |2-1| = 1,$$

   and $0 \neq 2$ yet $f(0) = f(2)$. Hence $f$ is not one-one.

   *Not onto:* The range of $f$ is $[0, \infty)$ because absolute value is nonnegative. Therefore negative real numbers (for example $-1$) are not attained. Thus $f$ is not onto $\mathbb{R}$.

   **Verification:** The example 0 and 2 shows failure of injectivity; the absence of negative values shows failure of surjectivity.

8. **Answer:** $\boxed{\text{Local maximum at } x = \tfrac{1}{3} \text{ with } f\big(\tfrac{1}{3}\big) = \tfrac{4}{27}; \quad \text{Local minimum at } x = 1 \text{ with } f(1) = 0.}$

   **Solution:** Let $f(x) = x(x-1)^2$. Compute derivative:

$$f'(x) = (x-1)^2 + x \cdot 2(x-1) = (x-1)\big((x-1) + 2x\big) = (x-1)(3x-1).$$

   Critical points: $f'(x) = 0$ gives $x = 1$ and $x = \frac{1}{3}$.

   Use first-derivative sign test:

   - For $x < \frac{1}{3}$, pick $x = 0$: $f'(0) = (-1)(-1) = +1 > 0$ (increasing).
   - For $\frac{1}{3} < x < 1$, pick $x = \frac{1}{2}$: $f'(\frac{1}{2}) = (-\frac{1}{2})(\frac{1}{2}) = -\frac{1}{4} < 0$ (decreasing).
   - For $x > 1$, pick $x = 2$: $f'(2) = (1)(5) = 5 > 0$ (increasing).

   Thus at $x = \frac{1}{3}$ we have $+ \to -$ so a *local maximum*; at $x = 1$ we have $- \to +$ so a *local minimum*.

   Values:

$$f\big(\tfrac{1}{3}\big) = \frac{1}{3}\Big(-\frac{2}{3}\Big)^2 = \frac{1}{3} \cdot \frac{4}{9} = \frac{4}{27}, \qquad f(1) = 1 \cdot 0^2 = 0.$$

   **Verification:** Sign changes of $f'$ confirm local max at $1/3$ and local min at 1; computed function values match the answer.

9. **Answer:** $\boxed{\text{Yes. } P(A \cap B) = 0.24 = P(A)P(B), \text{ so } A \text{ and } B \text{ are independent.}}$

   **Solution:** Two events $A$ and $B$ are independent iff $P(A \cap B) = P(A)P(B)$. Given $P(A) = 0.6$, $P(B) = 0.4$,

$$P(A)P(B) = 0.6 \times 0.4 = 0.24,$$

   which equals the given $P(A \cap B) = 0.24$. Hence $A$ and $B$ are independent.

   **Verification:** Direct multiplication yields the stated value 0.24.

10. **Answer:** $\boxed{k = 0.5}$.

   **Solution:** The probabilities must sum to 1:

$$0.3 + k + 0.2 = 1 \quad \Rightarrow \quad k = 1 - 0.5 = 0.5.$$

   **Verification:** $0.3 + 0.5 + 0.2 = 1$. Correct.

# Question 2 (3 × 2 Marks = 6 Marks)

1. **Answer:**

$$\boxed{\frac{dy}{dx} = \frac{(\cos x)^x\big(\ln(\cos x) - x\tan x\big) + x^y \cdot \frac{y}{x}}{1 - x^y \ln x}}$$

   (provided $x > 0$ so $\ln x$ is defined and the denominator $1 - x^y \ln x \neq 0$).

   **Solution:** Given

$$y = (\cos x)^x + x^y.$$

   Differentiate both sides w.r.t. $x$. Use the formula for differentiating $u^v$ when both $u$ and $v$ depend on $x$:

$$\frac{d}{dx}(u^v) = u^v\Big(v' \ln u + v\frac{u'}{u}\Big).$$

   For the first term $u = \cos x,\ v = x$:

$$\frac{d}{dx}(\cos x)^x = (\cos x)^x\Big(1 \cdot \ln(\cos x) + x \cdot \frac{-\sin x}{\cos x}\Big) = (\cos x)^x\big(\ln(\cos x) - x\tan x\big).$$

   For the second term $x^y$ where $y = y(x)$:

$$\frac{d}{dx}x^y = x^y\Big(y' \ln x + y \cdot \frac{1}{x}\Big).$$

   Differentiate the equation:

$$y' = (\cos x)^x\big(\ln(\cos x) - x\tan x\big) + x^y\big(y' \ln x + \tfrac{y}{x}\big).$$

Group terms with $y'$:

$$y' - x^y \ln x \, y' = (\cos x)^x \big( \ln(\cos x) - x \tan x \big) + x^y \cdot \frac{y}{x}.$$

Hence

$$y' \big( 1 - x^y \ln x \big) = (\cos x)^x \big( \ln(\cos x) - x \tan x \big) + x^y \cdot \frac{y}{x},$$

and finally

$$y' = \frac{(\cos x)^x \big( \ln(\cos x) - x \tan x \big) + x^y \cdot \frac{y}{x}}{1 - x^y \ln x}.$$

**Verification:** Substituting this $y'$ into the differentiated equation yields identity, provided denominator nonzero. This completes the derivation.

2. **Answer:** $\boxed{\text{There exists } c = 2.5 \in (1,4) \text{ such that } f'(c) = \frac{f(4) - f(1)}{4 - 1} = 1.}$

   **Solution:** The function $f(x) = x^2 - 4x - 3$ is a polynomial, hence continuous on $[1, 4]$ and differentiable on $(1, 4)$. Compute

$$\frac{f(4) - f(1)}{4 - 1} = \frac{(-3) - (-6)}{3} = \frac{3}{3} = 1.$$

   Compute derivative:

$$f'(x) = 2x - 4.$$

   Set $f'(c) = 1$ to find the required $c$:

$$2c - 4 = 1 \quad \Rightarrow \quad 2c = 5 \quad \Rightarrow \quad c = \frac{5}{2} = 2.5,$$

   which lies in $(1, 4)$. This verifies Lagrange's Mean Value Theorem for the given function and interval.

   **Verification:** $f'(2.5) = 2(2.5) - 4 = 5 - 4 = 1$, which equals the secant slope computed above.

3. **Answer:** $\boxed{(P(A), P(B)) = \left(\tfrac{1}{2}, \tfrac{2}{3}\right) \text{ or } \left(\tfrac{1}{3}, \tfrac{1}{2}\right).}$

   **Solution:** Let $p = P(A)$ and $q = P(B)$. Independence is assumed, and we are given

$$P(A \cap B') = p(1 - q) = \frac{1}{6}, \qquad P(A' \cap B) = (1 - p)q = \frac{1}{3}.$$

   From the first equation:

$$p - pq = \frac{1}{6} \quad \Rightarrow \quad pq = p - \frac{1}{6}.$$

   From the second:

$$q - pq = \frac{1}{3} \quad \Rightarrow \quad pq = q - \frac{1}{3}.$$

Equate the two expressions for $pq$:

$$p - \frac{1}{6} = q - \frac{1}{3} \quad \Rightarrow \quad p - q = -\frac{1}{6} \quad \Rightarrow \quad p = q - \frac{1}{6}.$$

Substitute $p = q - \frac{1}{6}$ into $p(1-q) = \frac{1}{6}$:

$$(q - \tfrac{1}{6})(1-q) = \tfrac{1}{6}.$$

Multiply out:

$$q - q^2 - \tfrac{1}{6} + \tfrac{q}{6} = \tfrac{1}{6}.$$

Multiply by 6:

$$6q - 6q^2 - 1 + q = 1 \quad \Rightarrow \quad -6q^2 + 7q - 2 = 0.$$

Multiply by $-1$:

$$6q^2 - 7q + 2 = 0.$$

Solve quadratic: discriminant $\Delta = (-7)^2 - 4 \cdot 6 \cdot 2 = 49 - 48 = 1$. Thus

$$q = \frac{7 \pm 1}{12} \quad \Rightarrow \quad q = \frac{8}{12} = \frac{2}{3} \text{ or } q = \frac{6}{12} = \frac{1}{2}.$$

Corresponding $p = q - \frac{1}{6}$ gives

$$\begin{cases} q = \frac{2}{3} \ \Rightarrow \ p = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}, \\ q = \frac{1}{2} \ \Rightarrow \ p = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}. \end{cases}$$

Both pairs satisfy the given equations and are valid probability values in $[0,1]$.

**Verification:** Check the first pair $(p,q) = (\frac{1}{2}, \frac{2}{3})$:

$$p(1-q) = \tfrac{1}{2} \cdot \tfrac{1}{3} = \tfrac{1}{6}, \qquad (1-p)q = \tfrac{1}{2} \cdot \tfrac{2}{3} = \tfrac{1}{3}.$$

Check the second pair $(p,q) = (\frac{1}{3}, \frac{1}{2})$:

$$p(1-q) = \tfrac{1}{3} \cdot \tfrac{1}{2} = \tfrac{1}{6}, \qquad (1-p)q = \tfrac{2}{3} \cdot \tfrac{1}{2} = \tfrac{1}{3}.$$

Both work. Thus the two solutions are correct.

# Question 3 ($4 \times 4 = 16$ marks)

1. **Answer:** $\boxed{\frac{1}{a}}$.

   **Solution:** Given the parametric curve

$$x = a(\theta - \sin\theta), \qquad y = a(1 + \cos\theta).$$

Compute derivatives with respect to $\theta$:

$$\frac{dx}{d\theta} = a(1-\cos\theta), \qquad \frac{dy}{d\theta} = -a\sin\theta.$$

Thus

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-\cos\theta)} = -\frac{\sin\theta}{1-\cos\theta} = -\cot\frac{\theta}{2},$$

(using $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ and $1-\cos\theta = 2\sin^2\frac{\theta}{2}$).
Differentiate $dy/dx = -\cot(\theta/2)$ with respect to $\theta$:

$$\frac{d}{d\theta}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\big(-\cot(\tfrac{\theta}{2})\big) = \tfrac{1}{2}\csc^2\big(\tfrac{\theta}{2}\big).$$

Therefore the second derivative with respect to $x$ is

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{d\theta}} = \frac{\frac{1}{2}\csc^2\big(\frac{\theta}{2}\big)}{a(1-\cos\theta)} = \frac{\frac{1}{2}\csc^2\big(\frac{\theta}{2}\big)}{2a\sin^2\big(\frac{\theta}{2}\big)} = \frac{1}{4a\sin^4\big(\frac{\theta}{2}\big)}.$$

Now evaluate at $\theta = \dfrac{\pi}{2}$. Then $\dfrac{\theta}{2} = \dfrac{\pi}{4}$, $\sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$, so

$$\sin^4\big(\tfrac{\pi}{4}\big) = \left(\frac{\sqrt{2}}{2}\right)^4 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Hence

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/2} = \frac{1}{4a\cdot\frac{1}{4}} = \frac{1}{a}.$$

**Verification:** The algebraic simplifications are consistent; final value $1/a$ is correct.

2. **Answer:** $\boxed{\text{Tangent: } y+4 = \tfrac{3}{4}(x-3); \qquad \text{Normal: } y+4 = -\tfrac{4}{3}(x-3).}$

   **Solution:** For the circle $x^2+y^2=25$, differentiate implicitly:

$$2x + 2y\frac{dy}{dx} = 0 \quad\Rightarrow\quad \frac{dy}{dx} = -\frac{x}{y}.$$

At the point $(3,-4)$,

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$$

So the tangent line has slope $m_T = \frac{3}{4}$ and equation

$$y-(-4) = \frac{3}{4}(x-3) \quad\Rightarrow\quad y+4 = \frac{3}{4}(x-3).$$

The normal has slope $m_N = -\dfrac{1}{m_T} = -\dfrac{4}{3}$ and equation

$$y + 4 = -\frac{4}{3}(x - 3).$$

**Verification:** The tangent slope $\frac{3}{4}$ satisfies the circle's normal condition; both equations pass through $(3, -4)$.

3. **Answer:** $\boxed{0}$.

   **Solution:** Let
$$I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx.$$
   Use the substitution $x \mapsto \dfrac{\pi}{2} - x$. Note $\sin\bigl(\frac{\pi}{2} - x\bigr) = \cos x$, $\cos\bigl(\frac{\pi}{2} - x\bigr) = \sin x$, so
$$I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx = -\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx = -I.$$
   Hence $I = -I$, which implies $I = 0$.

   (Alternatively one may note the integrand is an odd function with respect to the midpoint $\pi/4$ under this symmetry.)

   **Verification:** The symmetry argument is valid on $[0, \pi/2]$; therefore the integral equals 0.

4. **Answer:** $\boxed{-(a-b)(b-c)(c-a)(a+b+c)}$.

   **Solution:**

   *(i) Determinant sign change under row interchange.* Let $A$ be any square matrix and let $B$ be obtained from $A$ by interchanging two rows (say $R_1 \leftrightarrow R_2$). The determinant is an alternating multilinear function of the rows; in particular swapping two rows changes the sign of the determinant. A short justification: if $D$ denotes the determinant as a multilinear alternating map, then swapping those two rows multiplies the permutation associated to each term by a transposition, which changes the sign of every permutation, hence $|B| = -|A|$. Thus $R_1 \leftrightarrow R_2 \implies |B| = -|A|$.

   *(ii) Evaluate the given determinant after row swap.* We are given
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$
   Swapping the first two rows produces
$$\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ a^3 & b^3 & c^3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a+b+c).$$

**Verification:** The row-swap property gives the negative of the given expression, which matches the boxed answer.

## Question 4 ($3 \times 6 = 18$ marks)

1. **Answer:** $\boxed{\text{Semi-vertical angle } \alpha = \tan^{-1}(\sqrt{2}).}$

   **Solution:** Let the fixed slant length be $l$. For a right circular cone with semi-vertical angle $\alpha$,

   $$\text{radius } r = l\sin\alpha, \qquad \text{height } h = l\cos\alpha.$$

   The volume of the cone is

   $$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(l\sin\alpha)^2(l\cos\alpha) = \frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha.$$

   To maximize $V$ for fixed $l$, maximize the factor

   $$f(\alpha) = \sin^2\alpha\cos\alpha \qquad (0 < \alpha < \tfrac{\pi}{2}).$$

   Differentiate:

   $$f'(\alpha) = 2\sin\alpha\cos\alpha\cos\alpha + \sin^2\alpha(-\sin\alpha) = \sin\alpha\big(2\cos^2\alpha - \sin^2\alpha\big).$$

   Set $f'(\alpha) = 0$. Excluding the trivial $\sin\alpha = 0$, we get

   $$2\cos^2\alpha - \sin^2\alpha = 0 \quad \Rightarrow \quad 2(1-\sin^2\alpha) - \sin^2\alpha = 0$$

   $$\Rightarrow \quad 2 - 3\sin^2\alpha = 0 \quad \Rightarrow \quad \sin^2\alpha = \frac{2}{3}.$$

   Thus

   $$\tan^2\alpha = \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2,$$

   so $\tan\alpha = \sqrt{2}$ and

   $$\alpha = \tan^{-1}(\sqrt{2}).$$

   A second derivative check (or inspection) shows this critical point gives a maximum (endpoints give zero volume).

   **Verification:** The calculus yields $\tan\alpha = \sqrt{2}$, i.e. $\alpha = \tan^{-1}(\sqrt{2})$, as required.

2. **Answer:** $\boxed{\ln|x-1| - 5\ln|x-2| + 4\ln|x-3| + C.}$

   **Solution:** Evaluate

   $$\int \frac{3x-1}{(x-1)(x-2)(x-3)}\,dx.$$

Use partial fractions:

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Multiply through by $(x-1)(x-2)(x-3)$ and equate:

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Plugging convenient values:

For $x=1$: $3(1)-1=2=A(-1)(-2)=2A \Rightarrow A=1$.
For $x=2$: $6-1=5=B(1)(-1)=-B \Rightarrow B=-5$.
For $x=3$: $9-1=8=C(2)(1)=2C \Rightarrow C=4$.

Thus

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{4}{x-3},$$

and integrating termwise gives

$$\int \frac{3x-1}{(x-1)(x-2)(x-3)}\,dx = \ln|x-1| - 5\ln|x-2| + 4\ln|x-3| + C.$$

**Verification:** Differentiating the result reproduces the integrand.

3. **Answer:** $\boxed{y(x) = x^2 + C\csc x\,,\ \text{where } C \text{ is an arbitrary constant (and } \sin x \neq 0).}$

**Solution:** Solve the linear first-order ODE

$$\frac{dy}{dx} + y\cot x = 2x + x^2\cot x.$$

This is in standard form $y' + P(x)y = Q(x)$ with $P(x) = \cot x$. The integrating factor is

$$\mu(x) = e^{\int \cot x\,dx} = e^{\ln(\sin x)} = \sin x \quad (\text{for } \sin x > 0; \text{ formula holds up to sign otherwise}).$$

Multiply the ODE by $\sin x$:

$$\sin x\frac{dy}{dx} + y\sin x\cot x = 2x\sin x + x^2\cos x.$$

But $\sin x\cot x = \cos x$, so the left-hand side is

$$\sin x\frac{dy}{dx} + y\cos x = \frac{d}{dx}\big(y\sin x\big).$$

Thus

$$\frac{d}{dx}\big(y\sin x\big) = 2x\sin x + x^2\cos x.$$

Integrate both sides:

$$y\sin x = \int \big(2x\sin x + x^2\cos x\big)\,dx.$$

Compute the integral by parts (or note the convenient cancellation):

$$\int 2x \sin x \, dx = -2x \cos x + 2 \sin x,$$

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx = x^2 \sin x - \big(-2x \cos x + 2 \sin x\big) = x^2 \sin x + 2x \cos x - 2 \sin x.$$

Adding gives

$$\int \big(2x \sin x + x^2 \cos x\big) \, dx = (-2x \cos x + 2 \sin x) + (x^2 \sin x + 2x \cos x - 2 \sin x) = x^2 \sin x + C.$$

Therefore

$$y \sin x = x^2 \sin x + C \quad \Rightarrow \quad y = x^2 + C \csc x,$$

where $C$ is an arbitrary constant.

**Verification:** Differentiate $y = x^2 + C \csc x$:

$$y' = 2x + C(-\csc x \cot x).$$

Compute $y' + y \cot x$:

$$2x - C \csc x \cot x + (x^2 + C \csc x) \cot x = 2x + x^2 \cot x,$$

so the solution satisfies the ODE.

## Question 5 (15 Marks)

1. **(a)** — **Answer:**

$\boxed{\text{The function } f(x) = \sin x \text{ on } \mathbb{R} \text{ is not invertible. Restrict the domain to } [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ (or any interval of}}$

$$\boxed{\text{Then } f : [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to [-1, 1] \text{ is bijective and } f^{-1} = \arcsin.}$$

**Solution:** A function is invertible if and only if it is one–one (injective) and onto (surjective). The sine function $f : \mathbb{R} \to [-1, 1]$ satisfies $\mathrm{Im}(f) = [-1, 1]$, so it is onto $[-1, 1]$. However, it is not one–one on $\mathbb{R}$, because there exist distinct $x_1, x_2 \in \mathbb{R}$ with $\sin x_1 = \sin x_2$. For example,

$$\sin 0 = 0 = \sin \pi, \quad 0 \neq \pi,$$

so $f(0) = f(\pi)$ and $f$ is not injective; hence not invertible on $\mathbb{R}$.

To make $f$ invertible, we must restrict its domain to an interval on which $\sin x$ is one–one and whose image is $[-1, 1]$. A standard choice is the principal branch $[-\frac{\pi}{2}, \frac{\pi}{2}]$. On this interval, $\sin x$ is strictly increasing (hence one–one) and the image is $[-1, 1]$. Therefore

$$f : [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to [-1, 1] \quad \text{is bijective,}$$

and the inverse is the arcsine function $f^{-1}(y) = \arcsin y$ for $y \in [-1, 1]$.

**Verification:** On $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\sin x$ is strictly increasing and attains all values in $[-1, 1]$, so it is bijective and invertible with inverse arcsin.

2. **(b) — Answer:**

$$\boxed{(x, y, z) = (-13, 6, -2)}.$$

**Solution (Matrix Inverse Method Explained and Applied):**

Write the system in matrix form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}.$$

If $A$ is invertible then $\mathbf{x} = A^{-1}\mathbf{b}$. Instead of computing $A^{-1}$ explicitly, we may use elimination, which is algebraically equivalent to forming $A^{-1}$. Here we show a direct elimination (this gives the same result as using $A^{-1}\mathbf{b}$).

From the first equation $x + 2y - z = 1$ we get

$$x = 1 - 2y + z.$$

Substitute into the second equation:

$$3(1 - 2y + z) + 8y + 2z = 5 \Rightarrow 3 - 6y + 3z + 8y + 2z = 5 \Rightarrow 2y + 5z = 2. \qquad (1)$$

Substitute $x = 1 - 2y + z$ into the third equation:

$$4(1 - 2y + z) + 9y - z = 4 \Rightarrow 4 - 8y + 4z + 9y - z = 4 \Rightarrow y + 3z = 0 \Rightarrow y = -3z. \quad (2)$$

Substitute (2) into (1):

$$2(-3z) + 5z = 2 \Rightarrow -6z + 5z = 2 \Rightarrow -z = 2 \Rightarrow z = -2.$$

Then $y = -3z = -3(-2) = 6$. Finally,

$$x = 1 - 2y + z = 1 - 12 - 2 = -13.$$

Thus $(x, y, z) = (-13, 6, -2)$.

**Verification:** Substitute into the original equations:

$$\begin{aligned} x + 2y - z &= -13 + 12 - (-2) = 1, \\ 3x + 8y + 2z &= -39 + 48 - 4 = 5, \\ 4x + 9y - z &= -52 + 54 + 2 = 4, \end{aligned}$$

all satisfied, so the solution is correct.

3. **(c) — Answer:**

$$\boxed{\frac{11}{243}}.$$

**Solution:** Each question has three choices with one correct answer, so a random guess has probability $p = \frac{1}{3}$ of being correct and $q = 1 - p = \frac{2}{3}$ of being wrong. For $n = 5$ independent questions, the number of correct answers $X$ is binomial $\mathrm{Bin}(5, \frac{1}{3})$.

We want $P(X \geq 4) = P(X = 4) + P(X = 5)$. Using the binomial formula,

$$P(X = k) = \binom{5}{k} \left(\tfrac{1}{3}\right)^k \left(\tfrac{2}{3}\right)^{5-k}.$$

Compute:

$$P(X = 4) = \binom{5}{4} \left(\tfrac{1}{3}\right)^4 \left(\tfrac{2}{3}\right) = 5 \cdot \frac{1}{81} \cdot \frac{2}{3} = \frac{10}{243},$$

$$P(X = 5) = \binom{5}{5} \left(\tfrac{1}{3}\right)^5 = 1 \cdot \frac{1}{243} = \frac{1}{243}.$$

Hence

$$P(X \geq 4) = \frac{10}{243} + \frac{1}{243} = \frac{11}{243}.$$

**Verification:** Numerical sum $10/243 + 1/243 = 11/243$ is the required probability.

# SECTION B

## Question 6 (5 Marks)

1. **(a) — Answer:**

$$\boxed{\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \frac{10}{6}\mathbf{b} = \frac{5}{3}\mathbf{b} = \Big(\frac{5}{3}, \frac{10}{3}, \frac{5}{3}\Big).}$$

**Solution:** For $\vec{a} = (2, 3, 2)$ and $\vec{b} = (1, 2, 1)$,

$$\vec{a} \cdot \vec{b} = 2 \cdot 1 + 3 \cdot 2 + 2 \cdot 1 = 2 + 6 + 2 = 10,$$

and $|\vec{b}|^2 = 1^2 + 2^2 + 1^2 = 6$. The vector projection of $\vec{a}$ on $\vec{b}$ is

$$\operatorname{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b} = \frac{10}{6}\vec{b} = \frac{5}{3}\vec{b}.$$

Writing components gives $\big(\frac{5}{3}, \frac{10}{3}, \frac{5}{3}\big)$.

**Verification:** Dotting the projection with $\vec{b}$ yields $(10/6)|\vec{b}|^2 = 10$, consistent with the scalar projection times $|\vec{b}|$.

2. **(b) — Answer:**

$$\boxed{\text{A vector of magnitude 9 perpendicular to both is } \pm(-3, 6, 6).}$$

**Solution:** If $\vec{a} = (4, -1, 3)$ and $\vec{b} = (-2, 1, -2)$, then any vector perpendicular to both is a scalar multiple of $\vec{a} \times \vec{b}$. Compute

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = \big((-1)(-2) - 3(1)\big)\mathbf{i} - \big(4(-2) - 3(-2)\big)\mathbf{j} + \big(4 \cdot 1 - (-1)(-2)\big)\mathbf{k}.$$

Simplifying gives

$$\vec{a} \times \vec{b} = (-1, 2, 2).$$

Its magnitude is

$$|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3.$$

To obtain magnitude 9, multiply by 3:

$$3(\vec{a} \times \vec{b}) = 3(-1, 2, 2) = (-3, 6, 6).$$

The opposite vector $(3, -6, -6)$ is also acceptable (same magnitude, opposite direction).

**Verification:**

$$(-3, 6, 6) \cdot (4, -1, 3) = -12 - 6 + 18 = 0,$$
$$(-3, 6, 6) \cdot (-2, 1, -2) = 6 + 6 - 12 = 0,$$

so the vector is perpendicular to both. Its magnitude $\sqrt{(-3)^2 + 6^2 + 6^2} = \sqrt{9 + 36 + 36} = \sqrt{81} = 9$. Hence correct.

## Question 7 (10 Marks)

**(1) — Answer:** $\boxed{51x + 15y - 50z + 173 = 0.}$

**Solution:**

The general plane through the intersection of $P_1 : x + 2y + 3z - 4 = 0$ and $P_2 : 2x + y - z + 5 = 0$ is $\Pi : (x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0, \quad \lambda \in \mathbb{R}.$

Normal vector of $\Pi$ is $\mathbf{n}_\Pi = (1 + 2\lambda,\, 2 + \lambda,\, 3 - \lambda).$

We require $\Pi \perp (5x + 3y + 6z + 8 = 0),$ so $\mathbf{n}_\Pi \cdot (5, 3, 6) = 0.$

$(1 + 2\lambda)5 + (2 + \lambda)3 + (3 - \lambda)6 = 0.$

$5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0 \;\Rightarrow\; 29 + 7\lambda = 0.$

$\therefore\; \lambda = -\dfrac{29}{7}.$

Substitute $\lambda$ into $\Pi$ :

$(1 + 2\lambda)x + (2 + \lambda)y + (3 - \lambda)z - 4 + 5\lambda = 0.$

Compute coefficients: $1 + 2\lambda = 1 - \dfrac{58}{7} = -\dfrac{51}{7}, \quad 2 + \lambda = 2 - \dfrac{29}{7} = -\dfrac{15}{7},$

$$3 - \lambda = 3 + \frac{29}{7} = \frac{50}{7}, \quad -4 + 5\lambda = -4 - \frac{145}{7} = -\frac{173}{7}.$$

Thus $-\frac{51}{7}x - \frac{15}{7}y + \frac{50}{7}z - \frac{173}{7} = 0$. Multiply by $-7$ to get

$$51x + 15y - 50z + 173 = 0.$$

**Verification:** The plane is of the form $P_1 + \lambda P_2$ with $\lambda = -\frac{29}{7}$, its normal $(51, 15, -50)$ satisfies $(51, 15, -50) \cdot (5, 3, 6) = 255 + 45 - 300 = 0$, hence perpendicular to $(5, 3, 6)$.

**(2) — Answer:** $\boxed{A = \dfrac{16\pi + 4\sqrt{3}}{3} = \dfrac{4}{3}(4\pi + \sqrt{3}).}$

**Solution:**

Region common to the circle $x^2 + y^2 = 16$ (radius 4) and the parabola $y^2 = 6x$.

Intersection points: substitute $x = \dfrac{y^2}{6}$ into $x^2 + y^2 = 16:$

$$\left(\frac{y^2}{6}\right)^2 + y^2 = 16 \;\Rightarrow\; \frac{y^4}{36} + y^2 - 16 = 0.$$

Let $t = y^2:\;\; t^2 + 36t - 576 = 0$. Discriminant $\Delta = 36^2 + 4 \times 576 = 3600$,

$$\sqrt{\Delta} = 60, \quad t = \frac{-36 \pm 60}{2} \Rightarrow t = 12 \text{ (discard } t = -48).$$

$$\therefore y^2 = 12 \Rightarrow y = \pm 2\sqrt{3}, \quad x = \frac{y^2}{6} = \frac{12}{6} = 2.$$

For $y \in [-2\sqrt{3}, 2\sqrt{3}],$ the right boundary is the circle $x = \sqrt{16 - y^2}$,

the left boundary is the parabola $x = \dfrac{y^2}{6}$.

Area $A = \displaystyle\int_{y=-2\sqrt{3}}^{2\sqrt{3}} \left(\sqrt{16 - y^2} - \frac{y^2}{6}\right) dy = 2\int_0^{2\sqrt{3}} \left(\sqrt{16 - y^2} - \frac{y^2}{6}\right) dy.$

Compute $I_1 = \displaystyle\int_0^{2\sqrt{3}} \sqrt{16 - y^2}\, dy.$

Use $\displaystyle\int \sqrt{a^2 - y^2}\, dy = \frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right).$

With $a = 4,\; y = 2\sqrt{3}:\;\; \sqrt{16 - y^2} = \sqrt{4} = 2.$

$$I_1 = \left[\frac{y}{2}\sqrt{16 - y^2} + 8\sin^{-1}\left(\frac{y}{4}\right)\right]_0^{2\sqrt{3}} = 2\sqrt{3} + 8\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= 2\sqrt{3} + 8 \times \frac{\pi}{3} = 2\sqrt{3} + \frac{8\pi}{3}.$$

Compute $I_2 = \displaystyle\int_0^{2\sqrt{3}} \frac{y^2}{6}\, dy = \frac{1}{6} \cdot \frac{y^3}{3}\bigg|_0^{2\sqrt{3}} = \frac{1}{18}(2\sqrt{3})^3 = \frac{1}{18} \times 24\sqrt{3} = \frac{4\sqrt{3}}{3}.$

Thus $A = 2(I_1 - I_2) = 2\left(2\sqrt{3} + \dfrac{8\pi}{3} - \dfrac{4\sqrt{3}}{3}\right) = 2\left(\dfrac{6\sqrt{3} - 4\sqrt{3}}{3} + \dfrac{8\pi}{3}\right)$

$$= 2\left(\frac{2\sqrt{3}}{3} + \frac{8\pi}{3}\right) = \frac{4\sqrt{3}}{3} + \frac{16\pi}{3} = \frac{16\pi + 4\sqrt{3}}{3} = \frac{4}{3}(4\pi + \sqrt{3}).$$

**Verification:** Intersections at $(2, \pm 2\sqrt{3})$ used correctly;
the integrand is even, and the integration of $\sqrt{16-y^2}$ and polynomial term are correct, yielding $A = \dfrac{16\pi + 4\sqrt{3}}{3}$.

# SECTION C (15 Marks)

# Question 8 (5 Marks)

*Answer the following questions.*

1. A manufacturing company determines that the marginal cost for its product is given by $MC = 6x + 5$ and the marginal revenue is $MR = 10$. The fixed cost is Rs. 10. Find the total profit function $P(x)$ and determine the number of units $x$ that maximizes the profit. Also, find the maximum profit. [5]

**Answer.**

$$\boxed{P(x) = 5x - 3x^2 - 10, \qquad P_{\max} \text{ occurs at } x = \frac{5}{6}, \qquad P_{\max} = -\frac{95}{12}\ (\approx -7.9167).}$$

**Detailed solution.**

**Step 1: Relate marginal profit to given marginal revenue and marginal cost.**

Marginal profit $MP(x) = \dfrac{dP}{dx} = MR - MC$. Given $MR = 10$ and $MC = 6x + 5$, so

$$\frac{dP}{dx} = MP(x) = 10 - (6x + 5) = 5 - 6x.$$

**Step 2: Integrate to get total profit function.**

Integrate $dP/dx = 5 - 6x$:

$$P(x) = \int (5 - 6x)\, dx = 5x - 3x^2 + C,$$

where $C$ is an integration constant. Use the fixed cost to find $C$.

**Step 3: Determine constant $C$ from fixed cost.**

Profit $P(x) = \text{Revenue}(x) - \text{Cost}(x)$. At $x = 0$ revenue is 0, cost equals fixed cost $= 10$, so profit at $x = 0$ is $P(0) = -10$. Thus

$$P(0) = 5 \cdot 0 - 3 \cdot 0^2 + C = -10 \Longrightarrow C = -10.$$

Therefore

$$\boxed{P(x) = 5x - 3x^2 - 10.}$$

**Step 4: Maximize $P(x)$.**

Differentiate:

$$P'(x) = 5 - 6x.$$

Set $P'(x) = 0$ for critical points:

$$5 - 6x = 0 \Longrightarrow x = \frac{5}{6}.$$

Second derivative:

$$P''(x) = -6 < 0,$$

so $x = \dfrac{5}{6}$ is a local (indeed global, since $P$ is a downward parabola) maximum.

**Step 5: Maximum profit.**

Evaluate $P$ at $x = \dfrac{5}{6}$:

$$P\left(\tfrac{5}{6}\right) = 5 \cdot \tfrac{5}{6} - 3 \cdot \left(\tfrac{5}{6}\right)^2 - 10 = \tfrac{25}{6} - 3 \cdot \tfrac{25}{36} - 10 = \tfrac{25}{6} - \tfrac{75}{36} - 10.$$

Bring to common denominator 36:

$$\tfrac{150}{36} - \tfrac{75}{36} - \tfrac{360}{36} = \tfrac{150-75-360}{36} = \tfrac{-285}{36} = -\frac{95}{12}.$$

So the maximum profit is $-\dfrac{95}{12} \approx -7.9167$ (i.e. a minimum loss).

**Verification / Check (solution vs answer).**

- The derived profit function $P(x) = 5x - 3x^2 - 10$ differentiates to $P'(x) = 5 - 6x$, which matches the marginal profit $MP = MR - MC = 5 - 6x$.
- Setting $P' = 0$ gives $x = 5/6$, which matches the answer.
- Evaluating $P$ at $x = 5/6$ yields $-\dfrac{95}{12}$, matching the stated maximum profit.

All computations are consistent; there is no error in the question (the negative "maximum" simply means the firm cannot make a positive profit under the given linear marginal cost and fixed cost — the least loss occurs at $x = 5/6$ units).

# Question 9 (10 Marks)

*Answer the following questions.*

(a) Solve the following Linear Programming Problem graphically:

$$\text{Maximize } Z = x + 2y$$

Subject to

$$\begin{aligned} x + y &\le 9, \\ 2x + y &\ge 4, \\ x + 2y &\ge 6, \\ x, y &\ge 0. \end{aligned}$$

[4]

**Answer.**

$$\boxed{\text{Maximum value } Z_{\max} = 18 \text{ at } (x, y) = (0, 9).}$$

**Detailed solution.**

**Step 1: Rewrite constraints as boundary lines.**

$$\begin{array}{ll} \text{(i)} & x + y = 9 \quad (y = 9 - x), \\ \text{(ii)} & 2x + y = 4 \quad (y = 4 - 2x), \\ \text{(iii)} & x + 2y = 6 \quad (y = \frac{6-x}{2}), \\ \text{(iv)} & x = 0,\ y = 0 \text{ (nonnegativity).} \end{array}$$

The feasible region is

$$\{(x, y) \mid y \le 9 - x,\ y \ge 4 - 2x,\ y \ge \tfrac{6-x}{2},\ x \ge 0,\ y \ge 0\}.$$

**Step 2: Find intersection points (candidate vertices).**

Compute pairwise intersections (only those in the first quadrant and satisfying all constraints are candidates):

$$2x + y = 4 \text{ and } x + 2y = 6 \implies (x, y) = \left(\tfrac{2}{3}, \tfrac{8}{3}\right).$$

$$2x + y = 4 \text{ and } x = 0 \implies (0, 4).$$

$$x + 2y = 6 \text{ and } y = 0 \implies (6, 0).$$

$$x + y = 9 \text{ and } x = 0 \implies (0, 9).$$

$$x + y = 9 \text{ and } y = 0 \implies (9, 0).$$

Each of these lies in the feasible region (verify by checking all inequalities). Thus the feasible polygon has the extreme points (among others) at

$$(0,4),\ \left(\tfrac{2}{3},\tfrac{8}{3}\right),\ (6,0),\ (9,0),\ (0,9).$$

**Step 3: Evaluate objective function at these vertices.**

$$\begin{aligned}
(0,4):&\quad Z = 0 + 2\cdot 4 = 8,\\
\left(\tfrac{2}{3},\tfrac{8}{3}\right):&\quad Z = \tfrac{2}{3} + 2\cdot\tfrac{8}{3} = \tfrac{2+16}{3} = 6,\\
(6,0):&\quad Z = 6 + 0 = 6,\\
(9,0):&\quad Z = 9 + 0 = 9,\\
(0,9):&\quad Z = 0 + 2\cdot 9 = 18.
\end{aligned}$$

**Step 4: Conclude maximum.** The largest value is $Z_{\max} = 18$ attained at $(x,y) = (0,9)$. Hence

$$\boxed{(x,y) = (0,9), \quad Z_{\max} = 18.}$$

**Verification / Check.** The point $(0,9)$ satisfies all constraints:

$$x + y = 9 \le 9, \quad 2x + y = 9 \ge 4, \quad x + 2y = 18 \ge 6, \quad x, y \ge 0,$$

and yields $Z = 18$, which is the maximum among corner values. Thus the solution is correct.

(b) The following data gives the marks obtained by 10 students in Mathematics ($x$) and Physics ($y$):

| $x$ | 35 | 40 | 30 | 50 | 45 | 25 | 60 | 55 | 42 | 38 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 40 | 50 | 35 | 55 | 60 | 30 | 65 | 58 | 45 | 40 |

Find the regression line of $y$ on $x$ and estimate the marks in Physics of a student who scored 48 marks in Mathematics. [6]

**Answer.**

$$\boxed{\hat{y} = a + bx, \quad b = \frac{179}{178} \approx 1.005618, \quad a = \frac{2476}{445} \approx 5.564045.}$$

$$\boxed{\text{Estimated physics marks for } x = 48: \ \hat{y}\Big|_{x=48} = \frac{23956}{445} \approx 53.8596\ (\approx 53.86).}$$

**Detailed solution.**

**Step 1: Compute basic sums (for $n = 10$).**

$$\begin{aligned} n &= 10, \\ \sum x_i &= 420, \\ \sum y_i &= 478, \\ \sum x_i^2 &= 18708, \\ \sum x_i y_i &= 21150. \end{aligned}$$

(These sums are obtained by straightforward addition of the table entries; intermediate arithmetic is shown below for clarity:

$$\sum x_i = 35 + 40 + 30 + 50 + 45 + 25 + 60 + 55 + 42 + 38 = 420,$$

$$\sum y_i = 40 + 50 + 35 + 55 + 60 + 30 + 65 + 58 + 45 + 40 = 478,$$

$$\sum x_i^2 = 35^2 + 40^2 + \cdots + 38^2 = 18708,$$

$$\sum x_i y_i = 35 \cdot 40 + 40 \cdot 50 + \cdots + 38 \cdot 40 = 21150.$$

)

**Step 2: Compute slope $b$ of regression line $y$ on $x$.**

Use

$$b = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2}.$$

Substitute:

$$b = \frac{10 \cdot 21150 - 420 \cdot 478}{10 \cdot 18708 - 420^2} = \frac{211500 - 200760}{187080 - 176400} = \frac{10740}{10680} = \frac{179}{178} \approx 1.005618.$$

**Step 3: Compute intercept $a$.**

The means:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{420}{10} = 42, \qquad \bar{y} = \frac{478}{10} = 47.8.$$

Then

$$a = \bar{y} - b\bar{x} = 47.8 - \frac{179}{178} \cdot 42.$$

Compute exactly:

$$\frac{179}{178} \cdot 42 = \frac{7518}{178} \approx 42.235955,$$

so

$$a = 47.8 - \frac{7518}{178} = \frac{2476}{445} \approx 5.564045.$$

Thus the regression line is

$$\boxed{\hat{y} = \frac{2476}{445} + \frac{179}{178} x \ \approx \ 5.5640 + 1.005618\, x.}$$

**Step 4: Estimate $y$ when $x = 48$.**

$$\hat{y}(48) = \frac{2476}{445} + \frac{179}{178} \cdot 48 = \frac{2476}{445} + \frac{4296}{89} = \frac{2476}{445} + \frac{21480}{445} = \frac{23956}{445} \approx 53.8596.$$

Hence the estimated Physics marks for a student scoring 48 in Mathematics is about 53.86.

**Verification / Check (solution vs answer).**

- The slope $b$ computed from the standard formula equals $\frac{179}{178}$ (approx. 1.0056). Re-computing the numerator and denominator gives 10740 and 10680 respectively, whose quotient reduces to 179/178 — consistent.
- The intercept $a = \bar{y} - b\bar{x}$ was evaluated exactly (and reduced to 2476/445) and numerically $\approx 5.5640$ — consistent.
- Plugging $x = 48$ into the regression line gives $\hat{y} = \frac{23956}{445} \approx 53.8596$, matching the numeric evaluation above.

All calculations are consistent and free of arithmetic mistakes.

**Remark.** The regression line $y$ on $x$ was found using the standard least squares (linear) formulae. All intermediate sums and algebraic simplifications are shown; fractions are reduced where convenient and numerical approximations are provided for clarity.

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**Summary of final answers (compact):**

**Q8** $P(x) = 5x - 3x^2 - 10$, maximum at $x = 5/6$, $P_{\max} = -\frac{95}{12}$ ($\approx -7.9167$).

**Q9(a)** $Z_{\max} = 18$ at $(x, y) = (0, 9)$.

**Q9(b)** Regression of $y$ on $x$: $\hat{y} = \frac{2476}{445} + \frac{179}{178}x \approx 5.5640 + 1.005618x$. For $x = 48$, $\hat{y} \approx 53.86$.