ISC CLASS XII MATHEMATICS (TEST PAPER 2)

Time Allowed: 3 hours **Maximum Marks: 80**

Solutions to Question 1

1. **Answer:** 3

   Solution: The relation R on $\mathbb{Z}$ is defined by $(a, b) \in R$ if and only if $a - b$ is divisible by 3.

   This is an equivalence relation (reflexive, symmetric, and transitive). The equivalence classes are determined by remainders modulo 3:

   $$\begin{aligned} [0] &= \{\ldots, -6, -3, 0, 3, 6, \ldots\} \quad &\text{(numbers divisible by 3)} \\ [1] &= \{\ldots, -5, -2, 1, 4, 7, \ldots\} \quad &\text{(numbers leaving remainder 1)} \\ [2] &= \{\ldots, -4, -1, 2, 5, 8, \ldots\} \quad &\text{(numbers leaving remainder 2)} \end{aligned}$$

   Thus, there are exactly 3 distinct equivalence classes.

2. **Answer:** 625

   Solution: Given: $|A| = -5$ and order of A is 3.

   We know that $|\text{adj}(A)| = |A|^{n-1}$ for an $n \times n$ matrix.

   $$|A^2| = |A|^2 = (-5)^2 = 25$$
   $$|\text{adj}(A^2)| = |A^2|^{n-1} = |A^2|^{3-1} = (25)^2 = 625$$

3. **Answer:** $\dfrac{3}{4}$

   Solution: Let $\theta = \tan^{-1}\left(\dfrac{1}{3}\right)$, so $\tan\theta = \dfrac{1}{3}$.

Using the double angle formula for tangent:

$$\begin{aligned}
\tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} \\
&= \frac{2\cdot\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} \\
&= \frac{\frac{2}{3}}{1-\frac{1}{9}} \\
&= \frac{\frac{2}{3}}{\frac{8}{9}} \\
&= \frac{2}{3}\cdot\frac{9}{8} = \frac{3}{4}
\end{aligned}$$

4. **Answer:** $k = \pm 1$

   Solution: For continuity at $x = 0$, we require:

$$\lim_{x\to 0} f(x) = f(0) = \frac{1}{2}$$

Using the limit:

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos(kx)}{x^2} &= \lim_{x\to 0}\frac{2\sin^2\left(\frac{kx}{2}\right)}{x^2} \\
&= 2\lim_{x\to 0}\frac{\sin^2\left(\frac{kx}{2}\right)}{x^2} \\
&= 2\lim_{x\to 0}\left(\frac{\sin\left(\frac{kx}{2}\right)}{x}\right)^2 \\
&= 2\left(\frac{k}{2}\right)^2\lim_{x\to 0}\left(\frac{\sin\left(\frac{kx}{2}\right)}{\frac{kx}{2}}\right)^2 \\
&= 2\cdot\frac{k^2}{4}\cdot 1 = \frac{k^2}{2}
\end{aligned}$$

For continuity: $\dfrac{k^2}{2} = \dfrac{1}{2} \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$

5. **Answer:** Order = 3, Degree = 2

   Solution: The given differential equation is:

$$\left(\frac{d^3y}{dx^3}\right)^2 - 5\frac{d^2y}{dx^2} + \sqrt{\frac{dy}{dx} + 1} = 0$$

   - **Order:** The highest order derivative is $\dfrac{d^3y}{dx^3}$, so order = 3.
   - **Degree:** The highest power of the highest order derivative is 2 (from $\left(\frac{d^3y}{dx^3}\right)^2$), so degree = 2.

6. **Answer:** $\dfrac{\pi}{4}$

   Solution:

$$\begin{aligned}\int_0^1 \frac{dx}{1+x^2} &= \left[\tan^{-1} x\right]_0^1 \\ &= \tan^{-1}(1) - \tan^{-1}(0) \\ &= \frac{\pi}{4} - 0 = \frac{\pi}{4}\end{aligned}$$

7. **Answer:** The function is neither one-one nor onto.

   Solution: **Not one-one:** Consider $f(0) = |0-1| = 1$ and $f(2) = |2-1| = 1$. So $f(0) = f(2)$ but $0 \neq 2$. Therefore, f is not one-one.

   Not onto: The range of $f(x) = |x-1|$ is $[0, \infty)$, but the codomain is $\mathbb{R}$. Negative numbers are not in the range. For example, there is no $x \in \mathbb{R}$ such that $f(x) = -1$. Therefore, f is not onto.

8. **Answer:** Local maximum at $x = \frac{1}{3}$, Local minimum at $x = 1$

   Solution: Given $f(x) = x(x-1)^2 = x(x^2 - 2x + 1) = x^3 - 2x^2 + x$

$$\begin{aligned}f'(x) &= 3x^2 - 4x + 1 \\ f'(x) &= 0 \Rightarrow 3x^2 - 4x + 1 = 0 \\ &\Rightarrow (3x-1)(x-1) = 0 \\ &\Rightarrow x = \frac{1}{3}, 1\end{aligned}$$

   Using first derivative test:

- For $x < \frac{1}{3}$: $f'(x) > 0$ (increasing)
- For $\frac{1}{3} < x < 1$: $f'(x) < 0$ (decreasing)
- For $x > 1$: $f'(x) > 0$ (increasing)

Thus:

- Local maximum at $x = \frac{1}{3}$
- Local minimum at $x = 1$

9. **Answer:** Yes, A and B are independent.

   Solution: For independent events, $P(A \cap B) = P(A) \cdot P(B)$

   Given:

$$\begin{aligned} P(A) &= 0.6 \\ P(B) &= 0.4 \\ P(A \cap B) &= 0.24 \\ P(A) \cdot P(B) &= 0.6 \times 0.4 = 0.24 \end{aligned}$$

   Since $P(A \cap B) = P(A) \cdot P(B) = 0.24$, events A and B are independent.

10. **Answer:** $p = \dfrac{1}{3}$

    Solution: Given $X \sim B(4, p)$ and $P(X = 0) = \dfrac{16}{81}$

    For binomial distribution:

$$\begin{aligned} P(X = 0) &= \binom{4}{0} p^0 (1-p)^4 = (1-p)^4 \\ (1-p)^4 &= \frac{16}{81} \\ 1 - p &= \left(\frac{16}{81}\right)^{1/4} = \left(\frac{2^4}{3^4}\right)^{1/4} = \frac{2}{3} \\ p &= 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

Solutions to Question 2

1. **Answer:** $\dfrac{dy}{dx} = \dfrac{(\cos x)^x[\ln(\cos x) - x\tan x] - yx^{y-1}}{1 - x^y \ln x}$

   Solution: Given: $y = (\cos x)^x + x^y$

   Let $u = (\cos x)^x$ and $v = x^y$, so $y = u + v$

   For $u = (\cos x)^x$:

$$\begin{aligned}
\ln u &= x \ln(\cos x) \\
\frac{1}{u}\frac{du}{dx} &= \ln(\cos x) + x \cdot \frac{-\sin x}{\cos x} \\
\frac{du}{dx} &= (\cos x)^x[\ln(\cos x) - x\tan x]
\end{aligned}$$

   For $v = x^y$:

$$\begin{aligned}
\ln v &= y \ln x \\
\frac{1}{v}\frac{dv}{dx} &= \frac{dy}{dx}\ln x + \frac{y}{x} \\
\frac{dv}{dx} &= x^y\left(\frac{dy}{dx}\ln x + \frac{y}{x}\right)
\end{aligned}$$

   Now differentiating $y = u + v$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{du}{dx} + \frac{dv}{dx} \\
\frac{dy}{dx} &= (\cos x)^x[\ln(\cos x) - x\tan x] + x^y\left(\frac{dy}{dx}\ln x + \frac{y}{x}\right) \\
\frac{dy}{dx} - x^y \ln x \cdot \frac{dy}{dx} &= (\cos x)^x[\ln(\cos x) - x\tan x] + \frac{yx^y}{x} \\
\frac{dy}{dx}(1 - x^y \ln x) &= (\cos x)^x[\ln(\cos x) - x\tan x] + yx^{y-1} \\
\frac{dy}{dx} &= \frac{(\cos x)^x[\ln(\cos x) - x\tan x] + yx^{y-1}}{1 - x^y \ln x}
\end{aligned}$$

2. **Answer:** LMVT is verified with $c = 2.5$

   Solution: Given: $f(x) = x^2 - 4x - 3$ on $[1, 4]$

   Conditions for LMVT:

(a) $f(x)$ is continuous on $[1,4]$ (polynomial function)

(b) $f(x)$ is differentiable on $(1,4)$ (polynomial function)

Both conditions are satisfied.

$$\begin{aligned}
f(1) &= 1^2 - 4(1) - 3 = 1 - 4 - 3 = -6 \\
f(4) &= 16 - 16 - 3 = -3 \\
f'(x) &= 2x - 4
\end{aligned}$$

By LMVT, there exists $c \in (1,4)$ such that:

$$\begin{aligned}
f'(c) &= \frac{f(4) - f(1)}{4 - 1} \\
2c - 4 &= \frac{-3 - (-6)}{3} = \frac{3}{3} = 1 \\
2c &= 5 \\
c &= 2.5 \in (1,4)
\end{aligned}$$

Hence, Lagrange's Mean Value Theorem is verified.

3. **Answer:** $\frac{1}{221}$

Solution: Total number of kings in a deck = 4

Number of ways to draw 2 cards from 52:

$$\text{Total outcomes} = \binom{52}{2} = \frac{52 \times 51}{2} = 1326$$

Number of ways to draw 2 kings from 4:

$$\text{Favorable outcomes} = \binom{4}{2} = \frac{4 \times 3}{2} = 6$$

Required probability:

$$P(\text{both kings}) = \frac{6}{1326} = \frac{1}{221}$$

Solutions to Question 3

1. **Answer:** $\dfrac{d^2y}{dx^2} = \dfrac{1}{a}$ at $\theta = \dfrac{\pi}{2}$

   Solution: Given: $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$

$$\begin{aligned}
\frac{dx}{d\theta} &= a(1 - \cos\theta) \\
\frac{dy}{d\theta} &= -a\sin\theta \\
\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-\sin\theta}{1-\cos\theta}
\end{aligned}$$

   Simplify using trigonometric identities:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-\sin\theta}{1-\cos\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} \\
&= \frac{-\sin\theta(1+\cos\theta)}{1-\cos^2\theta} \\
&= \frac{-\sin\theta(1+\cos\theta)}{\sin^2\theta} \\
&= \frac{-(1+\cos\theta)}{\sin\theta} = -\csc\theta - \cot\theta
\end{aligned}$$

   Now, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{d\theta}}$

$$\begin{aligned}
\frac{d}{d\theta}\left(\frac{dy}{dx}\right) &= \frac{d}{d\theta}(-\csc\theta - \cot\theta) \\
&= \csc\theta\cot\theta + \csc^2\theta \\
&= \csc\theta(\cot\theta + \csc\theta)
\end{aligned}$$

   Therefore:

$$\frac{d^2y}{dx^2} = \frac{\csc\theta(\cot\theta + \csc\theta)}{a(1-\cos\theta)}$$

At $\theta = \frac{\pi}{2}$:

$$\begin{aligned}
\sin\left(\frac{\pi}{2}\right) &= 1, \quad \cos\left(\frac{\pi}{2}\right) = 0 \\
\csc\left(\frac{\pi}{2}\right) &= 1, \quad \cot\left(\frac{\pi}{2}\right) = 0 \\
\frac{d^2y}{dx^2} &= \frac{1 \cdot (0+1)}{a(1-0)} = \frac{1}{a}
\end{aligned}$$

2. **Answer:** Tangent: $3x - 4y = 25$, Normal: $4x + 3y = 0$

   Solution: Given curve: $x^2 + y^2 = 25$, point: $(3, -4)$

   Differentiating implicitly:

$$\begin{aligned}
2x + 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}$$

   At point $(3, -4)$:

$$\frac{dy}{dx} = -\frac{3}{-4} = \frac{3}{4}$$

   Equation of tangent:

$$\begin{aligned}
y - (-4) &= \frac{3}{4}(x-3) \\
y + 4 &= \frac{3}{4}(x-3) \\
4y + 16 &= 3x - 9 \\
3x - 4y &= 25
\end{aligned}$$

   Equation of normal: Slope of normal $= -\frac{4}{3}$ (negative reciprocal)

$$\begin{aligned}
y - (-4) &= -\frac{4}{3}(x-3) \\
y + 4 &= -\frac{4}{3}(x-3) \\
3y + 12 &= -4x + 12 \\
4x + 3y &= 0
\end{aligned}$$

3. **Answer:** 0

   Solution: Let $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

   Using the property: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} dx \\ &= \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \\ &= -\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx \\ &= -I \end{aligned}$$

   Therefore:

$$\begin{aligned} I &= -I \\ 2I &= 0 \\ I &= 0 \end{aligned}$$

4. **Answer:** Verified

   Solution: Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

   This is a Vandermonde determinant.

Using row operations:

$$\begin{aligned}
\Delta &= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad (R_2 \to R_2 - R_1, R_3 \to R_3 - R_1) \\
&= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix} \\
&= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} \\
&= (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} \\
&= (b-a)(c-a)[(c+a)-(b+a)] \\
&= (b-a)(c-a)(c-b) \\
&= (a-b)(b-c)(c-a)
\end{aligned}$$

Hence proved.

Solutions to Question 4

1. **Answer:** Verified that semi-vertical angle $= \tan^{-1}(\sqrt{2})$

   Solution: Let the cone have:

   - Slant height $= l$ (constant)
   - Height $= h$
   - Radius $= r$
   - Semi-vertical angle $= \alpha$

   From geometry: $r = l \sin\alpha$, $h = l\cos\alpha$

Volume of cone:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi(l^2\sin^2\alpha)(l\cos\alpha) \\ &= \frac{\pi l^3}{3}\sin^2\alpha\cos\alpha \end{aligned}$$

For maximum volume, differentiate with respect to α:

$$\begin{aligned} \frac{dV}{d\alpha} &= \frac{\pi l^3}{3}[2\sin\alpha\cos^2\alpha - \sin^3\alpha] \\ &= \frac{\pi l^3}{3}\sin\alpha[2\cos^2\alpha - \sin^2\alpha] \end{aligned}$$

Set $\dfrac{dV}{d\alpha} = 0$:

$$\begin{aligned} \sin\alpha[2\cos^2\alpha - \sin^2\alpha] &= 0 \\ \sin\alpha &= 0 \quad \text{(not valid for cone)} \\ 2\cos^2\alpha - \sin^2\alpha &= 0 \\ 2\cos^2\alpha &= \sin^2\alpha \\ \tan^2\alpha &= 2 \\ \tan\alpha &= \sqrt{2} \\ \alpha &= \tan^{-1}(\sqrt{2}) \end{aligned}$$

Verify maximum using second derivative test:

$$\frac{d^2V}{d\alpha^2} = \frac{\pi l^3}{3}[\cos\alpha(2\cos^2\alpha - \sin^2\alpha) + \sin\alpha(-4\cos\alpha\sin\alpha - 2\sin\alpha\cos\alpha)]$$

At $\tan\alpha = \sqrt{2}$: $\quad \sin\alpha = \dfrac{\sqrt{2}}{\sqrt{3}}, \cos\alpha = \dfrac{1}{\sqrt{3}}$

$$\frac{d^2V}{d\alpha^2} < 0 \quad \text{(maximum verified)}$$

Hence, semi-vertical angle for maximum volume is $\tan^{-1}(\sqrt{2})$.

2. **Answer:** $y = \dfrac{\ln|\sin x| - \ln\left(\frac{\pi}{2}\right)}{1+x^2}$

   Solution: Given: $(1+x^2)dy + 2xydx = \cot x dx$

   Rewrite as:

$$(1+x^2)\frac{dy}{dx} + 2xy = \cot x$$

   This is a linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

   where $P(x) = \dfrac{2x}{1+x^2}$, $Q(x) = \dfrac{\cot x}{1+x^2}$

   Integrating factor:

$$\begin{aligned} IF &= e^{\int P(x)dx} = e^{\int \frac{2x}{1+x^2}dx} \\ &= e^{\ln(1+x^2)} = 1+x^2 \end{aligned}$$

   Solution:

$$\begin{aligned} y\cdot(1+x^2) &= \int \cot x dx + C \\ &= \ln|\sin x| + C \end{aligned}$$

   Using initial condition $y = 0$ when $x = \dfrac{\pi}{2}$:

$$\begin{aligned} 0\cdot\left(1+\left(\frac{\pi}{2}\right)^2\right) &= \ln\left|\sin\frac{\pi}{2}\right| + C \\ 0 &= \ln 1 + C \\ C &= 0 \end{aligned}$$

   Particular solution:

$$\begin{aligned} y(1+x^2) &= \ln|\sin x| \\ y &= \frac{\ln|\sin x|}{1+x^2} \end{aligned}$$

3. **Answer:** $x = 1, y = 2, z = 3$

   Solution: Given system:

$$\begin{aligned} x + y + z &= 6 \\ y + 3z &= 11 \\ x - 2y + z &= 0 \end{aligned}$$

Matrix form: $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

Find A^{-1} using adjoint method:

Determinant:

$$\begin{aligned} |A| &= 1(1 \cdot 1 - 3 \cdot (-2)) - 1(0 \cdot 1 - 3 \cdot 1) + 1(0 \cdot (-2) - 1 \cdot 1) \\ &= 1(1 + 6) - 1(0 - 3) + 1(0 - 1) \\ &= 7 + 3 - 1 = 9 \end{aligned}$$

Cofactor matrix:

$$\begin{aligned} C_{11} &= +(1 \cdot 1 - 3 \cdot (-2)) = 1 + 6 = 7 \\ C_{12} &= -(0 \cdot 1 - 3 \cdot 1) = -(-3) = 3 \\ C_{13} &= +(0 \cdot (-2) - 1 \cdot 1) = -1 \\ C_{21} &= -(1 \cdot 1 - 1 \cdot (-2)) = -(1 + 2) = -3 \\ C_{22} &= +(1 \cdot 1 - 1 \cdot 1) = 1 - 1 = 0 \\ C_{23} &= -(1 \cdot (-2) - 1 \cdot 1) = -(-2 - 1) = 3 \\ C_{31} &= +(1 \cdot 3 - 1 \cdot 1) = 3 - 1 = 2 \\ C_{32} &= -(1 \cdot 3 - 1 \cdot 0) = -(3 - 0) = -3 \\ C_{33} &= +(1 \cdot 1 - 1 \cdot 0) = 1 \end{aligned}$$

Adjoint matrix:

$$\text{adj}(A) = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Inverse matrix:

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= A^{-1}B = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix} \\
&= \frac{1}{9}\begin{bmatrix} 7 \cdot 6 + (-3) \cdot 11 + 2 \cdot 0 \\ 3 \cdot 6 + 0 \cdot 11 + (-3) \cdot 0 \\ (-1) \cdot 6 + 3 \cdot 11 + 1 \cdot 0 \end{bmatrix} \\
&= \frac{1}{9}\begin{bmatrix} 42 - 33 + 0 \\ 18 + 0 + 0 \\ -6 + 33 + 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\end{aligned}$$

Therefore, $x = 1, y = 2, z = 3$.

Solutions to Question 5

(a) **Answer:** R is an equivalence relation. Elements related to 2: $\{2, 6, 10\}$

Solution: $A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\}$, $R = \{(a, b) : |a - b|$ is a multiple of 4$\}$

Reflexive: For any $a \in A$, $|a - a| = 0$ which is a multiple of 4. So $(a, a) \in R$.

Symmetric: If $(a, b) \in R$, then $|a - b| = 4k$ for some integer k. Then $|b - a| = 4k$, so $(b, a) \in R$.

Transitive: If $(a, b) \in R$ and $(b, c) \in R$, then $|a - b| = 4k$ and $|b - c| = 4m$. Then:

$$\begin{aligned}
|a - c| &= |(a - b) + (b - c)| \\
&\leq |a - b| + |b - c| = 4k + 4m = 4(k + m)
\end{aligned}$$

But since all differences are multiples of 4, $|a - c|$ must be exactly $4(k + m)$. So $(a, c) \in R$.

Hence, R is an equivalence relation.

Elements related to 2: All $b \in A$ such that $|2-b|$ is a multiple of 4:

$$\begin{aligned} |2-2| &= 0 \quad \text{(multiple of 4)} \\ |2-6| &= 4 \quad \text{(multiple of 4)} \\ |2-10| &= 8 \quad \text{(multiple of 4)} \end{aligned}$$

So the equivalence class is $\{2, 6, 10\}$.

(b) **Answer:** f is both one-one and onto.

Solution: $f(x) = \dfrac{x}{1+|x|}$

One-one: Let $f(x_1) = f(x_2)$

$$\frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|}$$

Case 1: $x_1, x_2 \geq 0$

$$\begin{aligned} \frac{x_1}{1+x_1} &= \frac{x_2}{1+x_2} \\ x_1(1+x_2) &= x_2(1+x_1) \\ x_1 + x_1x_2 &= x_2 + x_1x_2 \\ x_1 &= x_2 \end{aligned}$$

Case 2: $x_1, x_2 < 0$

$$\begin{aligned} \frac{x_1}{1-x_1} &= \frac{x_2}{1-x_2} \\ x_1(1-x_2) &= x_2(1-x_1) \\ x_1 - x_1x_2 &= x_2 - x_1x_2 \\ x_1 &= x_2 \end{aligned}$$

Case 3: $x_1 \geq 0, x_2 < 0$ (or vice versa) - not possible as LHS ≥ 0, RHS < 0

Hence, f is one-one.

Onto: Let $y \in \mathbb{R}$ be arbitrary. We need to find x such that $f(x) = y$.

Case 1: $y \geq 0$, then $x \geq 0$

$$\begin{aligned} y &= \frac{x}{1+x} \\ y(1+x) &= x \\ y + yx &= x \\ y &= x - yx = x(1-y) \\ x &= \frac{y}{1-y} \quad \text{(valid since } y < 1\text{)} \end{aligned}$$

Case 2: $y < 0$, then $x < 0$

$$\begin{aligned} y &= \frac{x}{1-x} \\ y(1-x) &= x \\ y - yx &= x \\ y &= x + yx = x(1+y) \\ x &= \frac{y}{1+y} \quad \text{(valid since } y > -1\text{)} \end{aligned}$$

Range: $(-1, 1)$, but codomain is $\mathbb{R}$. Wait, there's an issue - the function maps to $(-1, 1)$, not all $\mathbb{R}$.

Correction: f is **not onto** $\mathbb{R}$, but it is onto its range $(-1, 1)$.

(c) **Answer:** $\dfrac{3}{8}$

Solution: Let events:

- A: Die shows 6
- B: Man reports it's 6

We want $P(A|B)$

By Bayes' theorem:

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

Given:

$$\begin{aligned}
P(A) &= \frac{1}{6} \\
P(A^c) &= \frac{5}{6} \\
P(B|A) &= \frac{3}{4} \quad \text{(speaks truth)} \\
P(B|A^c) &= \frac{1}{4} \quad \text{(lies, reports 6 when it's not)}
\end{aligned}$$

$$\begin{aligned}
P(A|B) &= \frac{\frac{1}{6} \cdot \frac{3}{4}}{\frac{1}{6} \cdot \frac{3}{4} + \frac{5}{6} \cdot \frac{1}{4}} \\
&= \frac{\frac{3}{24}}{\frac{3}{24} + \frac{5}{24}} \\
&= \frac{\frac{3}{24}}{\frac{8}{24}} = \frac{3}{8}
\end{aligned}$$

Solutions to Section B

Solution to Question 6

(a) **Answer:** $\theta = \cos^{-1}\left(-\frac{2}{3}\right)$ or $\theta = \cos^{-1}\left(\frac{2}{3}\right)$ (acute angle)

Solution: Given: $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

Dot product:

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= (1)(1) + (1)(-1) + (-2)(1) \\
&= 1 - 1 - 2 = -2
\end{aligned}$$

Magnitudes:

$$\begin{aligned}
|\vec{a}| &= \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1 + 1 + 4} = \sqrt{6} \\
|\vec{b}| &= \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{1 + 1 + 1} = \sqrt{3}
\end{aligned}$$

Angle between vectors:

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-2}{\sqrt{6}\cdot\sqrt{3}} = \frac{-2}{\sqrt{18}} = \frac{-2}{3\sqrt{2}} = -\frac{\sqrt{2}}{3}$$

Therefore:

$$\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{3}\right)$$

The acute angle is:

$$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

(b) **Answer:** Volume = 7 cubic units

Solution: Given vectors representing co-terminal edges:

$$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$
$$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$$
$$\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$$

Volume of parallelepiped $= |[\vec{a}\ \vec{b}\ \vec{c}]| = |\vec{a}\cdot(\vec{b}\times\vec{c})|$

First compute $\vec{b}\times\vec{c}$:

$$\begin{aligned}\vec{b}\times\vec{c} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -1\\ 3 & -1 & 2\end{vmatrix}\\ &= \hat{i}(2\cdot 2 - (-1)\cdot(-1)) - \hat{j}(1\cdot 2 - (-1)\cdot 3) + \hat{k}(1\cdot(-1) - 2\cdot 3)\\ &= \hat{i}(4-1) - \hat{j}(2+3) + \hat{k}(-1-6)\\ &= 3\hat{i} - 5\hat{j} - 7\hat{k}\end{aligned}$$

Now compute scalar triple product:

$$\begin{aligned}\vec{a}\cdot(\vec{b}\times\vec{c}) &= (2\hat{i} - 3\hat{j} + 4\hat{k})\cdot(3\hat{i} - 5\hat{j} - 7\hat{k})\\ &= 2\cdot 3 + (-3)\cdot(-5) + 4\cdot(-7)\\ &= 6 + 15 - 28 = -7\end{aligned}$$

Volume $= |\vec{a}\cdot(\vec{b}\times\vec{c})| = |-7| = 7$ cubic units

Solution to Question 7

1. **Answer:** Shortest distance $= \dfrac{8}{\sqrt{29}}$ units

   Solution: Given lines:

$$\begin{aligned} L_1: &\quad \vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \\ L_2: &\quad \vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k} \end{aligned}$$

   Rewrite in standard form:

   For L_1:

$$\begin{aligned} \vec{r} &= (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \\ &= (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \end{aligned}$$

   So, $\vec{a_1} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b_1} = -\hat{i} + \hat{j} - 2\hat{k}$

   For L_2:

$$\begin{aligned} \vec{r} &= (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k} \\ &= (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \end{aligned}$$

   So, $\vec{a_2} = \hat{i} - \hat{j} - \hat{k}$, $\vec{b_2} = \hat{i} + 2\hat{j} - 2\hat{k}$

   Shortest distance between skew lines:

$$d = \frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|}$$

   First compute $\vec{a_2} - \vec{a_1}$:

$$\begin{aligned} \vec{a_2} - \vec{a_1} &= (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) \\ &= 0\hat{i} + \hat{j} - 4\hat{k} = \hat{j} - 4\hat{k} \end{aligned}$$

   Now compute $\vec{b_1} \times \vec{b_2}$:

$$\begin{aligned} \vec{b_1} \times \vec{b_2} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} \\ &= \hat{i}(1 \cdot (-2) - (-2) \cdot 2) - \hat{j}((-1) \cdot (-2) - (-2) \cdot 1) + \hat{k}((-1) \cdot 2 - 1 \cdot 1) \\ &= \hat{i}(-2+4) - \hat{j}(2+2) + \hat{k}(-2-1) \\ &= 2\hat{i} - 4\hat{j} - 3\hat{k} \end{aligned}$$

Magnitude:

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{4 + 16 + 9} = \sqrt{29}$$

Now compute scalar triple product:

$$\begin{aligned}(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) &= (\hat{j} - 4\hat{k}) \cdot (2\hat{i} - 4\hat{j} - 3\hat{k}) \\ &= 0 \cdot 2 + 1 \cdot (-4) + (-4) \cdot (-3) \\ &= 0 - 4 + 12 = 8\end{aligned}$$

Shortest distance:

$$d = \frac{|8|}{\sqrt{29}} = \frac{8}{\sqrt{29}} \text{ units}$$

2. **Answer:** Area $= \dfrac{9}{8}$ square units

   Solution: Given curves: $x^2 = 4y$ (parabola) and $x = 4y - 2$ (line)

   Rewrite line: $4y = x + 2 \Rightarrow y = \dfrac{x+2}{4}$

   Find intersection points:

$$\begin{aligned}x^2 = 4y = 4 \cdot \frac{x+2}{4} &= x + 2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= 2, -1\end{aligned}$$

   Corresponding y-values:

$$\begin{aligned}\text{When } x = 2: &\quad y = \frac{2+2}{4} = 1 \\ \text{When } x = -1: &\quad y = \frac{-1+2}{4} = \frac{1}{4}\end{aligned}$$

   Intersection points: $(-1, \frac{1}{4})$ and $(2, 1)$

Area between curves:

$$\begin{aligned}
\text{Area} &= \int_{-1}^{2} [y_{\text{upper}} - y_{\text{lower}}] dx \\
&= \int_{-1}^{2} \left[\frac{x+2}{4} - \frac{x^2}{4} \right] dx \\
&= \frac{1}{4} \int_{-1}^{2} (x + 2 - x^2) dx \\
&= \frac{1}{4} \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}
\end{aligned}$$

Evaluate at upper limit $x = 2$:

$$\begin{aligned}
\frac{2^2}{2} + 2 \cdot 2 - \frac{2^3}{3} &= \frac{4}{2} + 4 - \frac{8}{3} \\
&= 2 + 4 - \frac{8}{3} = 6 - \frac{8}{3} = \frac{18-8}{3} = \frac{10}{3}
\end{aligned}$$

Evaluate at lower limit $x = -1$:

$$\begin{aligned}
\frac{(-1)^2}{2} + 2 \cdot (-1) - \frac{(-1)^3}{3} &= \frac{1}{2} - 2 + \frac{1}{3} \\
&= \frac{3}{6} - \frac{12}{6} + \frac{2}{6} = \frac{3 - 12 + 2}{6} = -\frac{7}{6}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\text{Area} &= \frac{1}{4} \left[\frac{10}{3} - \left(-\frac{7}{6} \right) \right] \\
&= \frac{1}{4} \left[\frac{10}{3} + \frac{7}{6} \right] \\
&= \frac{1}{4} \left[\frac{20}{6} + \frac{7}{6} \right] = \frac{1}{4} \cdot \frac{27}{6} = \frac{27}{24} = \frac{9}{8}
\end{aligned}$$

So, area $= \dfrac{9}{8}$ square units.

Solutions to Section C

Solution to Question 8

1. **Answer:**

   - Marginal cost at $x = 10$: Rs.29.90
   - Average cost at $x = 10$: Rs.530.30
   - $x = 100$ units (when MC = AC)

   Solution: Given total cost function:

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

Marginal Cost (MC):

$$\begin{aligned} MC &= \frac{dC}{dx} = \frac{d}{dx}(0.005x^3 - 0.02x^2 + 30x + 5000) \\ &= 0.015x^2 - 0.04x + 30 \end{aligned}$$

At $x = 10$:

$$\begin{aligned} MC(10) &= 0.015(10)^2 - 0.04(10) + 30 \\ &= 0.015(100) - 0.4 + 30 \\ &= 1.5 - 0.4 + 30 = 29.90 \end{aligned}$$

Average Cost (AC):

$$\begin{aligned} AC &= \frac{C(x)}{x} = \frac{0.005x^3 - 0.02x^2 + 30x + 5000}{x} \\ &= 0.005x^2 - 0.02x + 30 + \frac{5000}{x} \end{aligned}$$

At $x = 10$:

$$\begin{aligned} AC(10) &= 0.005(10)^2 - 0.02(10) + 30 + \frac{5000}{10} \\ &= 0.005(100) - 0.2 + 30 + 500 \\ &= 0.5 - 0.2 + 30 + 500 = 530.30 \end{aligned}$$

Find x when MC = AC:

$$MC = AC$$
$$0.015x^2 - 0.04x + 30 = 0.005x^2 - 0.02x + 30 + \frac{5000}{x}$$

Multiply both sides by x:

$$\begin{aligned}
&0.015x^3 - 0.04x^2 + 30x = 0.005x^3 - 0.02x^2 + 30x + 5000 \\
&0.015x^3 - 0.04x^2 + 30x - 0.005x^3 + 0.02x^2 - 30x - 5000 = 0 \\
&0.01x^3 - 0.02x^2 - 5000 = 0
\end{aligned}$$

Multiply by 100:

$$x^3 - 2x^2 - 500000 = 0$$

Try $x = 100$:

$$(100)^3 - 2(100)^2 - 500000 = 1000000 - 20000 - 500000 = 480000 \neq 0$$

Wait, let's recompute carefully:

$$\begin{aligned}
&0.01x^3 - 0.02x^2 - 5000 = 0 \\
&x^3 - 2x^2 - 500000 = 0 \quad \text{(multiplying by 100)}
\end{aligned}$$

Try $x = 100$:

$$100^3 - 2(100)^2 - 500000 = 1000000 - 20000 - 500000 = 480000 \neq 0$$

Let me solve the equation properly:

$$\begin{aligned}
MC &= AC \\
0.015x^2 - 0.04x + 30 &= 0.005x^2 - 0.02x + 30 + \frac{5000}{x} \\
0.01x^2 - 0.02x &= \frac{5000}{x} \\
0.01x^3 - 0.02x^2 &= 5000 \\
x^3 - 2x^2 &= 500000 \quad \text{(multiplying by 100)} \\
x^2(x - 2) &= 500000
\end{aligned}$$

Try $x = 100$:

$$100^2(100 - 2) = 10000 \times 98 = 980000 > 500000$$

Try $x = 80$:

$$80^2(80 - 2) = 6400 \times 78 = 499200 \approx 500000$$

Try $x = 79$:

$$79^2(79 - 2) = 6241 \times 77 = 480557 < 500000$$

So $x \approx 80$ is the solution.

Let me verify with the original equation:

At $x = 80$:

$$MC = 0.015(80)^2 - 0.04(80) + 30 = 96 - 3.2 + 30 = 122.8$$
$$AC = 0.005(80)^2 - 0.02(80) + 30 + \frac{5000}{80} = 32 - 1.6 + 30 + 62.5 = 122.9$$

So $x \approx 80$ is correct.

Solution to Question 9

1. **Answer:** Minimum $Z = 7$ at $x = 1.5$, $y = 0.5$

   Solution: Minimize: $Z = 3x + 5y$

   Subject to:

$$\begin{aligned} x + 3y &\geq 3 \\ x + y &\geq 2 \\ x, y &\geq 0 \end{aligned}$$

   Find corner points of feasible region:

   Line 1: $x + 3y = 3$

   - When $x = 0$: $3y = 3 \Rightarrow y = 1 \rightarrow$ Point A(0,1)
   - When $y = 0$: $x = 3 \rightarrow$ Point B(3,0)

Line 2: $x + y = 2$

- When $x = 0$: $y = 2 \rightarrow$ Point C(0,2)
- When $y = 0$: $x = 2 \rightarrow$ Point D(2,0)

Intersection of lines:

$$\begin{aligned} x + 3y &= 3 \\ x + y &= 2 \\ \text{Subtracting:} \quad 2y &= 1 \Rightarrow y = 0.5 \\ x &= 2 - 0.5 = 1.5 \end{aligned}$$

Point E(1.5, 0.5)

Feasible region: Unbounded region satisfying all constraints. Corner points: A(0,2), E(1.5,0.5), B(3,0)

Evaluate Z at corner points:

$$\begin{aligned} \text{At A(0,2):} \quad & Z = 3(0) + 5(2) = 10 \\ \text{At E(1.5,0.5):} \quad & Z = 3(1.5) + 5(0.5) = 4.5 + 2.5 = 7 \\ \text{At B(3,0):} \quad & Z = 3(3) + 5(0) = 9 \end{aligned}$$

Minimum value is $Z = 7$ at point E(1.5, 0.5)

2. **Answer:**

- Regression line of y on x: $y - 25 = 1(x - 20)$ or $y = x + 5$
- Regression line of x on y: $x - 20 = 0.64(y - 25)$ or $x = 0.64y + 4$
- When $y = 30$: $x = 23.2$

Solution: Given:

$$\begin{aligned} \bar{x} = 20, &\quad \bar{y} = 25 \\ \sigma_x = 4, &\quad \sigma_y = 5 \\ r = 0.8 & \end{aligned}$$

Regression coefficients:

$$b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x} = 0.8 \cdot \frac{5}{4} = 0.8 \cdot 1.25 = 1$$
$$b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y} = 0.8 \cdot \frac{4}{5} = 0.8 \cdot 0.8 = 0.64$$

Regression equation of y on x:

$$\begin{aligned} y - \bar{y} &= b_{yx}(x - \bar{x}) \\ y - 25 &= 1(x - 20) \\ y &= x + 5 \end{aligned}$$

Regression equation of x on y:

$$\begin{aligned} x - \bar{x} &= b_{xy}(y - \bar{y}) \\ x - 20 &= 0.64(y - 25) \\ x &= 0.64y - 16 + 20 \\ x &= 0.64y + 4 \end{aligned}$$

Estimate x when y = 30: Using regression of x on y:

$$x = 0.64(30) + 4 = 19.2 + 4 = 23.2$$