# ISC CLASS XII MATHEMATICS (TEST PAPER 12) - SET 12

**Time Allowed: 3 hours** **Maximum Marks: 80**

## SECTION A (Compulsory - 65 Marks)

*All questions in this section are compulsory. (R&F: 10, Algebra: 10, Calculus: 32, Probability: 13)*

### Question 1 (10 × 1 Mark = 10 Marks)

*Answer the following questions.*

1. Let $*$ be an operation on $\mathbb{R}$ defined by $a * b = a + b + 5$. Find the inverse of the element 10.

   **Answer:** The inverse of the element 10 is $-15$.

   **Solution:** Let $e$ be the identity element for the operation $*$ on $\mathbb{R}$. Then, for any $a \in \mathbb{R}$,

   $$a * e = a \implies a + e + 5 = a \implies e = -5.$$

   Let $b$ be the inverse of 10. Then,

   $$10 * b = e \implies 10 + b + 5 = -5 \implies b = -20 + 5 = -15.$$

   Therefore, the inverse of 10 is $-15$.

2. Evaluate: $\sec^2(\tan^{-1}(2))$.

   **Answer:** The value is 5.

   **Solution:** Let $\theta = \tan^{-1}(2)$. Then, $\tan\theta = 2$.

   $$\sec^2\theta = 1 + \tan^2\theta = 1 + 2^2 = 5.$$

   Therefore, $\sec^2(\tan^{-1}(2)) = 5$.

3. State the domain of the function $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$.

   **Answer:** The domain of $f(x)$ is $\mathbb{R} \setminus \{0\}$.

   **Solution:** The function $\tan^{-1}\left(\frac{1}{x}\right)$ is defined for all real numbers except where the argument $\frac{1}{x}$ is undefined, i.e., when $x = 0$. Therefore, the domain of $f(x)$ is $\mathbb{R} \setminus \{0\}$.

4. Let $R$ be a relation on $\mathbb{N}$ defined by $xRy$ if $x$ divides $y$. Is $R$ a symmetric relation? Justify.

   **Answer:** No, $R$ is not a symmetric relation.

   **Solution:** A relation $R$ is symmetric if $xRy \implies yRx$ for all $x, y \in \mathbb{N}$. Consider $x = 2$ and $y = 4$. Then, 2 divides 4, so $2R4$. However, 4 does not divide 2, so $4R2$ is false. Therefore, $R$ is not symmetric.

5. Find $\frac{dy}{dx}$ if $y = \sqrt{\sin x}$.

   **Answer:** $\frac{dy}{dx} = \frac{\cos x}{2\sqrt{\sin x}}$.

   **Solution:** Using the chain rule,

   $$\frac{dy}{dx} = \frac{d}{dx}(\sin x)^{1/2} = \frac{1}{2}(\sin x)^{-1/2} \cdot \cos x = \frac{\cos x}{2\sqrt{\sin x}}.$$

6. If $y = e^{3x} + 2e^{-3x}$, find $\frac{d^2y}{dx^2}$.

   **Answer:** $\frac{d^2y}{dx^2} = 9e^{3x} + 18e^{-3x}$.

   **Solution:** First, find the first derivative:

$$\frac{dy}{dx} = 3e^{3x} - 6e^{-3x}.$$

   Now, find the second derivative:

$$\frac{d^2y}{dx^2} = 9e^{3x} + 18e^{-3x}.$$

7. Write the integrating factor (I.F.) of the differential equation $\frac{dy}{dx} + \frac{1}{x}y = \sin x$.

   **Answer:** The integrating factor is $x$.

   **Solution:** The differential equation is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x) = \frac{1}{x}$. The integrating factor is given by:

$$\text{I.F.} = e^{\int P(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln|x|} = x.$$

8. Find the value of $\int_0^2 (x^2 + 1)\,dx$.

   **Answer:** The value of the integral is $\frac{14}{3}$.

   **Solution:**

$$\int_0^2 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_0^2 = \left(\frac{8}{3} + 2\right) - (0+0) = \frac{8}{3} + \frac{6}{3} = \frac{14}{3}.$$

9. If $P(A') = 0.7$, $P(B') = 0.6$, and $P(A \cup B) = 0.6$. Find $P(A \cap B)$.

   **Answer:** $P(A \cap B) = 0.1$.

   **Solution:** Given $P(A') = 0.7$, so $P(A) = 1 - 0.7 = 0.3$. Given $P(B') = 0.6$, so $P(B) = 1 - 0.6 = 0.4$. Using the formula for the union of two sets:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

   Substituting the known values:

$$0.6 = 0.3 + 0.4 - P(A \cap B) \implies P(A \cap B) = 0.7 - 0.6 = 0.1.$$

10. A random variable $X$ has values $0, 1, 2$ with probabilities $0.1, 0.5, 0.4$ respectively. Find the mean $E(X)$.

    **Answer:** The mean $E(X)$ is 1.3.

    **Solution:** The mean (expected value) of $X$ is calculated as:

$$E(X) = \sum x \cdot P(X = x) = 0 \cdot 0.1 + 1 \cdot 0.5 + 2 \cdot 0.4 = 0 + 0.5 + 0.8 = 1.3.$$

## Question 2 (3 $\times$ 2 Marks = 6 Marks)

*Answer the following questions.*

1. Use differentiation to approximate the change in the area of a circle if its radius changes from 10 cm to 10.1 cm.

   **Answer:** The approximate change in the area is $2\pi$ cm$^2$.

   **Solution:** The area of a circle is given by $A = \pi r^2$. The derivative of $A$ with respect to $r$ is:

$$\frac{dA}{dr} = 2\pi r.$$

   For $r = 10$ cm, $\frac{dA}{dr} = 20\pi$ cm. The change in radius, $\Delta r = 0.1$ cm. The approximate change in area is:

$$\Delta A \approx \frac{dA}{dr} \cdot \Delta r = 20\pi \cdot 0.1 = 2\pi \text{ cm}^2.$$

2. Find the equation of the normal to the curve $y^2 = 4x$ at the point $(1, 2)$.

**Answer:** The equation of the normal is $y = -x + 3$.

**Solution:** Differentiate $y^2 = 4x$ implicitly with respect to $x$:

$$2y\frac{dy}{dx} = 4 \implies \frac{dy}{dx} = \frac{2}{y}.$$

At the point $(1, 2)$, the slope of the tangent is $\frac{2}{2} = 1$. The slope of the normal is $-1$ (negative reciprocal). The equation of the normal is:

$$y - 2 = -1(x - 1) \implies y = -x + 3.$$

3. From a pack of 52 cards, 2 cards are drawn at random without replacement. Find the probability that both are aces.

**Answer:** The probability is $\frac{1}{221}$.

**Solution:** The number of ways to draw 2 aces from 4 aces is $\binom{4}{2} = 6$. The number of ways to draw 2 cards from 52 cards is $\binom{52}{2} = 1326$. The probability is:

$$\frac{6}{1326} = \frac{1}{221}.$$

## Question 3 (4 × 4 Marks = 16 Marks)

*Answer the following questions.*

1. Find the particular solution of the differential equation: $\frac{dy}{dx} = \frac{x+y}{x-y}$, given $y = 0$ when $x = 1$.

**Answer:** The particular solution is $x^2 + y^2 - 2xy - 2x = 0$.

**Solution:** Substitute $y = vx$ to get:

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}.$$

Separate variables and integrate:

$$\int \frac{1-v}{1+v^2}dv = \int \frac{1}{x}dx.$$

Solving gives:

$$\tan^{-1} v - \frac{1}{2}\ln(1 + v^2) = \ln x + C.$$

Substitute $v = \frac{y}{x}$ and apply the initial condition $y(1) = 0$ to find $C = 0$. The solution simplifies to:

$$x^2 + y^2 - 2xy - 2x = 0.$$

2. Find the equation of the tangent and normal to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point $(1, 3)$.

**Answer:** The tangent is $y = 3x$, and the normal is $y = -\frac{1}{3}x + \frac{10}{3}$.

**Solution:** The derivative is:

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10.$$

At $x = 1$, $\frac{dy}{dx} = 3$. The tangent is:

$$y - 3 = 3(x - 1) \implies y = 3x.$$

The normal is:

$$y - 3 = -\frac{1}{3}(x - 1) \implies y = -\frac{1}{3}x + \frac{10}{3}.$$

3. Evaluate: $\int \frac{x^2+1}{x^4+x^2+1} dx$.

   **Answer:** The integral evaluates to $\tan^{-1}\left(\frac{x^2+1}{x}\right) + C$.

   **Solution:** Substitute $u = x - \frac{1}{x}$:

$$\int \frac{x^2+1}{x^4+x^2+1} dx = \int \frac{du}{u^2+1} = \tan^{-1} u + C = \tan^{-1}\left(\frac{x^2+1}{x}\right) + C.$$

4. Check if the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is involutory.

   **Answer:** Yes, $A$ is involutory.

   **Solution:** A matrix is involutory if $A^2 = I$.

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

   Therefore, $A$ is involutory.

## Question 4 (3 × 6 Marks = 18 Marks)

*Answer the following questions.*

1. Prove that $\begin{vmatrix} 1 & \alpha & \alpha^2 - \beta\gamma \\ 1 & \beta & \beta^2 - \gamma\alpha \\ 1 & \gamma & \gamma^2 - \alpha\beta \end{vmatrix} = 0$.

   **Solution:** Expand the determinant:

$$\begin{vmatrix} 1 & \alpha & \alpha^2 - \beta\gamma \\ 1 & \beta & \beta^2 - \gamma\alpha \\ 1 & \gamma & \gamma^2 - \alpha\beta \end{vmatrix} = 1 \cdot \begin{vmatrix} \beta & \beta^2 - \gamma\alpha \\ \gamma & \gamma^2 - \alpha\beta \end{vmatrix} - \alpha \cdot \begin{vmatrix} 1 & \beta^2 - \gamma\alpha \\ 1 & \gamma^2 - \alpha\beta \end{vmatrix} + (\alpha^2 - \beta\gamma) \cdot \begin{vmatrix} 1 & \beta \\ 1 & \gamma \end{vmatrix}.$$

   Simplify each minor:

$$= \beta(\gamma^2 - \alpha\beta) - \gamma(\beta^2 - \gamma\alpha) - \alpha(\gamma^2 - \alpha\beta - \beta^2 + \gamma\alpha) + (\alpha^2 - \beta\gamma)(\gamma - \beta).$$

   All terms cancel out, so the determinant is 0.

2. Show that a right circular cylinder of a given surface area and maximum volume is such that its height is equal to the diameter of the base.

   **Solution:** Let $r$ be the radius and $h$ the height. The surface area is:

$$S = 2\pi r^2 + 2\pi rh.$$

   The volume is:

$$V = \pi r^2 h.$$

   Express $h$ in terms of $r$ and $S$:

$$h = \frac{S - 2\pi r^2}{2\pi r}.$$

   Substitute into $V$ and maximize:

$$V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r}\right) = \frac{rS}{2} - \pi r^3.$$

   Differentiate $V$ with respect to $r$ and set to zero:

$$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2 = 0 \implies r = \sqrt{\frac{S}{6\pi}}.$$

   Substitute back to find $h = 2r$.

3. Evaluate: $\int_0^\pi \frac{x}{a^2\cos^2 x+b^2\sin^2 x}dx$.

   **Answer:** The integral evaluates to $\frac{\pi^2}{2ab}$.

   **Solution:** Use the substitution $u = \pi - x$:

   $$I = \int_0^\pi \frac{x}{a^2\cos^2 x + b^2\sin^2 x}dx = \int_0^\pi \frac{\pi - u}{a^2\cos^2 u + b^2\sin^2 u}du.$$

   Add the original and substituted integrals:

   $$2I = \pi \int_0^\pi \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx.$$

   Simplify using trigonometric identities:

   $$I = \frac{\pi}{2}\int_0^\pi \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx = \frac{\pi^2}{2ab}.$$

## Question 5 (15 Marks)

*Answer the following questions.*

1. **(a)** Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x - 5$ is invertible. Find its inverse function $f^{-1}$.

   **Answer:** The inverse function is $f^{-1}(x) = \frac{x+5}{4}$.

   **Solution:** To show invertibility, solve $y = 4x - 5$ for $x$:

   $$x = \frac{y+5}{4}.$$

   Thus, $f^{-1}(x) = \frac{x+5}{4}$.

2. **(b)** A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of getting at least one failure.

   **Answer:** The probability is $\frac{217}{216}$.

   **Solution:** The probability of a doublet is $\frac{6}{36} = \frac{1}{6}$. The probability of at least one failure is:

   $$1 - \left(\frac{1}{6}\right)^4 = \frac{217}{216}.$$

3. **(c)** Let $A$ and $B$ be two independent events. If $P(A) = 0.3$ and $P(B) = 0.4$, find the probability of occurrence of at least one of $A$ and $B$.

   **Answer:** The probability is 0.58.

   **Solution:** The probability of at least one of $A$ or $B$ is:

   $$P(A \cup B) = P(A) + P(B) - P(A)P(B) = 0.3 + 0.4 - (0.3)(0.4) = 0.58.$$

# SECTION B (Optional - 15 Marks)

*Answer all questions from this section. (Unit V: Vectors - 5 Marks; Unit VI: 3D Geometry - 6 Marks; Unit VII: Applications of Integrals - 4 Marks)*

## Question 6 (5 Marks)

*Answer the following questions.*

1. If $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} - \hat{\jmath} + 2\hat{k}$, find the magnitude of the vector $\vec{a} \times \vec{b}$.

   **Answer:** The magnitude of $\vec{a} \times \vec{b}$ is $\sqrt{74}$.

   **Solution:** The cross product $\vec{a} \times \vec{b}$ is given by:

   $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{\imath}(3 \cdot 2 - 1 \cdot (-1)) - \hat{\jmath}(2 \cdot 2 - 1 \cdot 1) + \hat{k}(2 \cdot (-1) - 3 \cdot 1)$$

   $$= \hat{\imath}(6 + 1) - \hat{\jmath}(4 - 1) + \hat{k}(-2 - 3) = 7\hat{\imath} - 3\hat{\jmath} - 5\hat{k}.$$

   The magnitude is:

   $$|\vec{a} \times \vec{b}| = \sqrt{7^2 + (-3)^2 + (-5)^2} = \sqrt{49 + 9 + 25} = \sqrt{83}.$$

   **Correction:** The magnitude is $\sqrt{83}$.

2. The position vectors of the vertices of a triangle are $\vec{a}, \vec{b}, \vec{c}$. Find the area of the triangle using the formula $\frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$.

   **Answer:** The area of the triangle is $\frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$.

   **Solution:** The area of the triangle formed by vectors $\vec{a}, \vec{b}, \vec{c}$ is:

   $$\text{Area} = \frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|.$$

   This formula is derived from the fact that the sum of the cross products represents twice the area of the triangle.

## Question 7 (10 Marks)

*Answer the following questions.*

1. Find the equation of the plane containing the lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and $\frac{x}{1} = \frac{y-2}{-2} = \frac{z-1}{3}$.

   **Answer:** The equation of the plane is $11x + 2y - 7z = 0$.

   **Solution:** Let the lines be $L_1$ and $L_2$. The direction vectors of $L_1$ and $L_2$ are $\vec{d_1} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and $\vec{d_2} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$. The normal vector $\vec{n}$ to the plane is $\vec{d_1} \times \vec{d_2}$:

   $$\vec{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 4 \\ 1 & -2 & 3 \end{vmatrix} = \hat{\imath}(-1 \cdot 3 - 4 \cdot (-2)) - \hat{\jmath}(2 \cdot 3 - 4 \cdot 1) + \hat{k}(2 \cdot (-2) - (-1) \cdot 1)$$

   $$= \hat{\imath}(-3 + 8) - \hat{\jmath}(6 - 4) + \hat{k}(-4 + 1) = 5\hat{\imath} - 2\hat{\jmath} - 3\hat{k}.$$

   Using point $(1, -1, 3)$ from $L_1$, the plane equation is:

   $$5(x - 1) - 2(y + 1) - 3(z - 3) = 0 \implies 5x - 2y - 3z = 0.$$

   **Correction:** The correct normal vector is $11\hat{\imath} + 2\hat{\jmath} - 7\hat{k}$. The correct plane equation is:

   $$11(x - 1) + 2(y + 1) - 7(z - 3) = 0 \implies 11x + 2y - 7z = 0.$$

2. Using integration, find the area bounded by the curve $y = |x - 1|$, the $x$-axis, and the lines $x = 0$ and $x = 2$.

   **Answer:** The area is 1.

   **Solution:** The curve $y = |x - 1|$ intersects the $x$-axis at $x = 1$. The area is the sum of two integrals:

$$\text{Area} = \int_0^1 (1 - x)\, dx + \int_1^2 (x - 1)\, dx$$
$$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2$$
$$= \left(1 - \frac{1}{2}\right) + \left(2 - 2 - \frac{1}{2} + 1\right) = \frac{1}{2} + \frac{1}{2} = 1.$$

# SECTION C (Optional - 15 Marks)

*Answer all questions from this section. (Unit VIII: Application of Calculus - 5 Marks; Unit IX: Linear Regression - 6 Marks; Unit X: Linear Programming - 4 Marks)*

## Question 8 (5 Marks)

*Answer the following question.*

1. The total cost function is $C(x) = 2x^3 - 15x^2 + 36x + 8$. Find the number of units $x$ for which the total cost is minimum.

   **Answer:** The total cost is minimum at $x = 2$ units.

   **Solution:** To find the minimum cost, first find the derivative of $C(x)$:

$$C'(x) = \frac{d}{dx}(2x^3 - 15x^2 + 36x + 8) = 6x^2 - 30x + 36.$$

   Set the derivative equal to zero to find critical points:

$$6x^2 - 30x + 36 = 0 \implies x^2 - 5x + 6 = 0 \implies (x - 2)(x - 3) = 0.$$

   The critical points are $x = 2$ and $x = 3$. To determine which point gives the minimum cost, find the second derivative:

$$C''(x) = 12x - 30.$$

   Evaluate $C''(x)$ at $x = 2$ and $x = 3$:

$$C''(2) = 24 - 30 = -6 < 0 \quad \text{(Local maximum)}$$
$$C''(3) = 36 - 30 = 6 > 0 \quad \text{(Local minimum)}$$

   Therefore, the total cost is minimum at $x = 3$ units. **Correction:** The total cost is minimum at $x = 3$ units.

## Question 9 (10 Marks)

*Answer the following questions.*

1. Solve the following Linear Programming Problem graphically: Minimize $Z = 3x + 5y$ Subject to the constraints:

$$x + 3y \geq 3$$
$$x + y \geq 2$$
$$x, y \geq 0$$

**Answer:** The minimum value of $Z$ is 7 at the point $(0, 3)$.

**Solution:** Plot the constraints:

- $x + 3y = 3$ intersects the axes at $(3, 0)$ and $(0, 1)$.
- $x + y = 2$ intersects the axes at $(2, 0)$ and $(0, 2)$.

The feasible region is the area satisfying all constraints. Evaluate $Z = 3x + 5y$ at the corner points:

- At $(0, 3)$: $Z = 3(0) + 5(3) = 15$
- At $(0, 2)$: $Z = 3(0) + 5(2) = 10$
- At $(1.5, 0.5)$: $Z = 3(1.5) + 5(0.5) = 4.5 + 2.5 = 7$
- At $(3, 0)$: $Z = 3(3) + 5(0) = 9$

The minimum value of $Z$ is 7 at the point $(1.5, 0.5)$. **Correction:** The minimum value of $Z$ is 7 at the point $(1.5, 0.5)$.

2. The regression equations are $3x + 2y = 26$ and $6x + y = 31$. Find the mean of $x$ and $y$ and the coefficient of correlation $r$. Assume the first equation is the regression line of $y$ on $x$.

**Answer:** The mean of $x$ is 4, the mean of $y$ is 7, and the coefficient of correlation $r$ is $-0.5$.

**Solution:** The regression lines are:

$$3x + 2y = 26 \quad \text{(Regression line of } y \text{ on } x\text{)}$$

$$6x + y = 31 \quad \text{(Regression line of } x \text{ on } y\text{)}$$

Solve the system of equations to find the means $\bar{x}$ and $\bar{y}$:

$$\begin{cases} 3\bar{x} + 2\bar{y} = 26 \\ 6\bar{x} + \bar{y} = 31 \end{cases}$$

Solving gives $\bar{x} = 4$ and $\bar{y} = 7$. The coefficient of correlation $r$ is given by:

$$r = \pm\sqrt{b_{yx} \cdot b_{xy}},$$

where $b_{yx} = -\frac{3}{2}$ and $b_{xy} = -\frac{1}{6}$.

$$r = -\sqrt{\left(-\frac{3}{2}\right)\left(-\frac{1}{6}\right)} = -\sqrt{\frac{1}{4}} = -\frac{1}{2}.$$