ISC CLASS XII MATHEMATICS (TEST PAPER 19) - SET 19

Time Allowed: 3 hours **Maximum Marks: 80**

SECTION A (Compulsory - 65 Marks)

All questions in this section are compulsory. (R&F: 10, Algebra: 10, Calculus: 32, Probability: 13)

Question 1 (10×1 Mark $= 10$ Marks)

Answer the following questions.

1. Let $*$ be a binary operation on $\mathbb{R}$ defined by $a * b = |a - b|$. Check if $*$ is associative. [1]
   Answer: $*$ is **not associative**.
   Solution: To be associative we would need
   $$(a * b) * c = a * (b * c) \quad \text{for all } a, b, c \in \mathbb{R}.$$
   Compute the two sides:
   $$(a * b) * c = \big(|a - b|\big) * c = \big||a - b| - c\big|, \qquad a * (b * c) = a * \big(|b - c|\big) = \big|a - |b - c|\big|.$$
   These expressions are not equal for all choices of a, b, c. For example, take $a = 10,\ b = 3,\ c = 1$:
   $$(a * b) * c = \big||10 - 3| - 1\big| = |7 - 1| = 6, \qquad a * (b * c) = \big|10 - |3 - 1|\big| = |10 - 2| = 8,$$
   so the two sides differ. Hence $*$ is not associative.

2. Evaluate: $\sin\big(\cos^{-1} \frac{3}{5}\big)$. [1]
   Answer: $\dfrac{4}{5}$.
   Solution: Let $\theta = \cos^{-1}\big(\frac{3}{5}\big)$, so $\cos\theta = \frac{3}{5}$ and $\theta \in [0, \pi]$. Then
   $$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \Big(\frac{3}{5}\Big)^2} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5},$$
   and sign is positive because $\theta \in [0, \pi]$ (where $\sin\theta \geq 0$). Thus $\sin(\cos^{-1}(3/5)) = \frac{4}{5}$.

3. State the domain of the function $f(x) = \sin^{-1}(2x^2 + 1)$. [1]
   Answer: The domain is $\{0\}$ (only $x = 0$).
   Solution: The principal inverse sine $\sin^{-1}(\cdot)$ is defined only for inputs in $[-1, 1]$. So we require
   $$-1 \leq 2x^2 + 1 \leq 1.$$
   Subtract 1: $-2 \leq 2x^2 \leq 0$, hence $-1 \leq x^2 \leq 0$. Since $x^2 \geq 0$ always, the only possibility is $x^2 = 0$, i.e. $x = 0$. Therefore the domain is $\{0\}$. (At $x = 0$, $2x^2 + 1 = 1$ and $\sin^{-1}(1) = \pi/2$.)

4. If $f : \{1, 2, 3\} \to \{a, b, c\}$ is given by $f(1) = a,\ f(2) = b,\ f(3) = c$, write the inverse function f^{-1}. [1]
   Answer: $f^{-1}(a) = 1,\ f^{-1}(b) = 2,\ f^{-1}(c) = 3$.

Solution: Since f is a bijection pairing $1 \mapsto a,\ 2 \mapsto b,\ 3 \mapsto c$, the inverse map reverses these pairs:

$$f^{-1} : \{a, b, c\} \to \{1, 2, 3\}, \qquad f^{-1}(a) = 1,\ f^{-1}(b) = 2,\ f^{-1}(c) = 3.$$

This is the required inverse.

5. Find $\dfrac{dy}{dx}$ if $y = \sqrt{\log(\sin x)}$. [1]

Answer: $\dfrac{dy}{dx} = \dfrac{\cot x}{2\sqrt{\ln(\sin x)}}$, valid where $\sin x > 1$???[1]

Solution: Interpret log as natural logarithm ln. Then

$$y = \big(\ln(\sin x)\big)^{1/2}.$$

Differentiate using chain rule:

$$\frac{dy}{dx} = \frac{1}{2}\big(\ln(\sin x)\big)^{-1/2} \cdot \frac{d}{dx}\big(\ln(\sin x)\big) = \frac{1}{2\sqrt{\ln(\sin x)}} \cdot \frac{\cos x}{\sin x}.$$

Hence

$$\boxed{\frac{dy}{dx} = \frac{\cot x}{2\sqrt{\ln(\sin x)}}.}$$

Domain remark: The expression is meaningful only when $\ln(\sin x) > 0$, i.e. $\sin x > 1$, which never occurs for real x. If the intended domain is where the expression under the outer square root is nonnegative, we require $\ln(\sin x) \geq 0 \iff \sin x \geq 1$, which forces $\sin x = 1$ (at isolated points $x = \pi/2 + 2k\pi$). At such isolated points the derivative is not defined (division by zero). Therefore the given function $y = \sqrt{\ln(\sin x)}$ has no open interval of real x where it is real-valued; the expression is complex for most real x.

Alternative interpretation: If the question intended $y = \sqrt{\ln|\sin x|}$ (or $y = (\ln(\sin x))^{1/2}$ with complex values), the formal derivative formula above holds algebraically, but for a real-valued function one usually needs $\sin x > 1$ which is impossible. Thus the original question likely intended a different inner function (for example $y = \sqrt{\ln(\sin x)}$ is not real-valued on any open interval). No arithmetic error in differentiation; the issue is with the real-valued domain of the given function.

6. Write the value of $\displaystyle\int_0^{\pi} \cos x\, dx$. [1]

Answer: 0.

Solution: An antiderivative of $\cos x$ is $\sin x$. Thus

$$\int_0^{\pi} \cos x\, dx = \big[\sin x\big]_0^{\pi} = \sin \pi - \sin 0 = 0 - 0 = 0.$$

Hence the value is 0.

7. Write the order and degree of the differential equation

$$\frac{d^2y}{dx^2} = \left(y + \left(\frac{dy}{dx}\right)^2\right)^{1/3}.$$

[1]

Answer: Order = 2. Degree: **not defined** (because the equation is not a polynomial in the derivatives).

Solution: The highest derivative present is $\dfrac{d^2y}{dx^2}$, so the order is 2. The degree of a differential equation is defined as the power (positive integer) of the highest order derivative after the

[1] See corrected domain note in solution.

equation has been made a polynomial in derivatives (i.e., clearing fractional powers). Here the highest derivative appears to the first power on the left, but the right-hand side involves a fractional power $(\cdots)^{1/3}$ of an expression containing lower order derivatives. One cannot algebraically transform the equation into a polynomial equation in $\frac{d^2y}{dx^2}$ with integer (non-fractional) exponents without introducing fractional powers. Therefore the degree is not defined.

8. Determine the value of k for which

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \le 2, \\ k & \text{if } x > 2 \end{cases}$$

is continuous at $x = 2$. [1]

Answer: $k = 3$.

Solution: For continuity at $x = 2$ we need $\lim_{x\to 2} f(x) = f(2)$. The left-hand limit (and the value at 2) is

$$f(2) = 2 \cdot 2 - 1 = 3.$$

The right-hand limit as $x \downarrow 2$ equals k. Equating, $k = 3$. Thus k must be 3 for continuity.

9. If $P(A \mid B) = P(A)$, what is the relationship between events A and B? [1]

Answer: A and B are **independent** (provided $P(B) > 0$).

Solution: By definition of conditional probability,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{(provided } P(B) > 0\text{)}.$$

If $P(A \mid B) = P(A)$ then

$$\frac{P(A \cap B)}{P(B)} = P(A) \quad \Longrightarrow \quad P(A \cap B) = P(A)P(B),$$

which is precisely the condition for independence of A and B. Thus A and B are independent (assuming $P(B) > 0$).

10. If A is a square matrix, show that $(A^T)^T = A$. [1]

Answer: $(A^T)^T = A$.

Solution: Let $A = (a_{ij})$ be an $n \times n$ matrix. The transpose A^T has entries $(A^T)_{ij} = a_{ji}$. Taking transpose again,

$$\left(A^T\right)^T_{ij} = (A^T)_{ji} = a_{ij},$$

so every entry of $(A^T)^T$ equals the corresponding entry of A. Hence $(A^T)^T = A$. This argument works for any rectangular matrix as well (not only square matrices).

Question 2 (3 × 2 Marks = 6 Marks)

Answer the following questions.

1. If $x = at^2$ and $y = 2at$, find $\frac{d^2y}{dx^2}$. [2]

Answer: $\frac{d^2y}{dx^2} = -\frac{1}{2a\,t^3} = -\frac{\sqrt{a}}{2\,x^{3/2}}$.

Solution: Differentiate with respect to t:

$$\frac{dx}{dt} = 2at, \qquad \frac{dy}{dt} = 2a.$$

Hence

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}.$$

Differentiate again using $\dfrac{d}{dx} = \dfrac{1}{dx/dt}\dfrac{d}{dt}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{\frac{d}{dt}\left(t^{-1}\right)}{dx/dt} = \frac{-t^{-2}}{2at} = -\frac{1}{2a\,t^3}.$$

If one prefers the result in terms of x, use $t = \sqrt{x/a}$ to obtain

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{a}}{2\,x^{3/2}}.$$

2. The radius of a cone is 4 cm and its height is 3 cm. If the height is increasing at a rate $dh/dt = 0.1$ cm/s while the radius is kept constant, find dV/dt. [2]

   Answer: $\dfrac{dV}{dt} = \dfrac{8}{15}\pi$ cm^3/s.

   Solution: Volume of a cone: $V = \dfrac{1}{3}\pi r^2 h$. With $r = 4$ fixed,

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt} = \frac{1}{3}\pi(4^2)(0.1) = \frac{1}{3}\pi\cdot 16\cdot 0.1 = \frac{16}{30}\pi = \frac{8}{15}\pi.$$

3. A bag contains 4 white and 6 black balls. Two balls are drawn without replacement. Find the probability that the second ball drawn is black. [2]

   Answer: $\dfrac{3}{5}$.

   Solution: Let B_2 denote the event "second ball is black." Using total probability,

$$P(B_2) = P(\text{first black, second black}) + P(\text{first white, second black}).$$

   Compute:

$$P(\text{BB}) = \frac{6}{10}\cdot\frac{5}{9} = \frac{30}{90} = \frac{1}{3}, \qquad P(\text{WB}) = \frac{4}{10}\cdot\frac{6}{9} = \frac{24}{90} = \frac{4}{15}.$$

   Thus

$$P(B_2) = \frac{1}{3} + \frac{4}{15} = \frac{5+4}{15} = \frac{9}{15} = \frac{3}{5}.$$

   (Alternatively, by symmetry the probability the second ball is black equals the initial fraction of black balls, $6/10 = 3/5$.)

Question 3 (4 × 4 Marks = 16 Marks)

Answer the following questions.

1. Find the equation of the tangent and normal to the curve $y = x^3 - 2x^2 + 4$ at the point $(2, 4)$. [4]

   Answer: Tangent: $y = 4x - 4$. Normal: $y = -\dfrac{1}{4}x + \dfrac{9}{2}$.

   Solution: Differentiate:

$$\frac{dy}{dx} = 3x^2 - 4x.$$

At $x = 2$, slope $m = \ 3(4) - 8 = 12 - 8 = 4$. Tangent through $(2, 4)$:

$$y - 4 = 4(x - 2) \quad \Longrightarrow \quad y = 4x - 4.$$

Normal slope $m_n = -\dfrac{1}{m} = -\dfrac{1}{4}$, so normal:

$$y - 4 = -\frac{1}{4}(x - 2) \quad \Longrightarrow \quad y = -\frac{1}{4}x + \frac{9}{2}.$$

2. Find the particular solution of the differential equation

$$x\frac{dy}{dx} = y + 2\sqrt{x^2 + y^2}, \qquad y(1) = 0.$$

[4]

Answer: $y(x) = x\,\sinh\bigl(2\ln x\bigr) \;=\; \dfrac{x^3 - x^{-1}}{2}.$

Solution: Put $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substitute:

$$x\bigl(v + x\frac{dv}{dx}\bigr) = vx + 2\sqrt{x^2 + v^2x^2} = vx + 2x\sqrt{1 + v^2}.$$

Cancel vx and divide by x:

$$x\frac{dv}{dx} = 2\sqrt{1 + v^2} \quad \Longrightarrow \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{2\,dx}{x}.$$

Integrate:

$$\sinh^{-1} v = 2\ln x + C.$$

Use $y(1) = 0 \Rightarrow v(1) = 0$ so $0 = 2\ln 1 + C$, hence $C = 0$. Thus

$$v = \sinh(2\ln x), \qquad y = xv = x\,\sinh(2\ln x).$$

Using the identity $\sinh(2\ln x) = \frac{1}{2}(e^{2\ln x} - e^{-2\ln x}) = \frac{1}{2}(x^2 - x^{-2})$, we get

$$y = \frac{x^3 - x^{-1}}{2},$$

which indeed satisfies $y(1) = 0$.

3. Evaluate: $\displaystyle\int \frac{x}{(x + 1)(x^2 + 1)}\,dx.$ [4]

Answer: $-\dfrac{1}{2}\ln|x + 1| + \dfrac{1}{4}\ln\bigl(x^2 + 1\bigr) + \dfrac{1}{2}\tan^{-1} x + C.$

Solution: Use partial fractions:

$$\frac{x}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}.$$

Equating numerators gives $x = A(x^2 + 1) + (Bx + C)(x + 1)$. Comparing coefficients yields $A = -\frac{1}{2},\ B = \frac{1}{2},\ C = \frac{1}{2}$. Hence

$$\frac{x}{(x + 1)(x^2 + 1)} = -\frac{1}{2}\cdot\frac{1}{x + 1} + \frac{1}{2}\cdot\frac{x}{x^2 + 1} + \frac{1}{2}\cdot\frac{1}{x^2 + 1}.$$

Integrate termwise:

$$\int \frac{x}{(x + 1)(x^2 + 1)}dx = -\frac{1}{2}\ln|x + 1| + \frac{1}{4}\ln\bigl(x^2 + 1\bigr) + \frac{1}{2}\tan^{-1} x + C.$$

4. If $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$, solve for matrix X if $2A - 3X = B$. [4]

   Answer: $X = \frac{1}{3}(2A - B) = \begin{pmatrix} 1 & 2 \\ -\frac{1}{3} & 0 \end{pmatrix}$.

   Solution: Rearranging gives $3X = 2A - B$, hence $X = \dfrac{2A - B}{3}$. Compute

$$2A = \begin{pmatrix} 4 & 6 \\ 2 & 4 \end{pmatrix}, \qquad 2A - B = \begin{pmatrix} 4-1 & 6-0 \\ 2-3 & 4-4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -1 & 0 \end{pmatrix}.$$

   Therefore

$$X = \frac{1}{3}\begin{pmatrix} 3 & 6 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -\frac{1}{3} & 0 \end{pmatrix}.$$

Question 4 (3 × 6 Marks = 18 Marks)

Answer the following questions.

1. Show that

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1 + pxyz)\,(x-y)(y-z)(z-x).$$

   [6]

   Answer: The identity holds; the determinant equals $(1 + pxyz)\,(x-y)(y-z)(z-x)$.

   Solution: Split the third column:

$$\det\begin{pmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{pmatrix} = \det\begin{pmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{pmatrix} + p\det\begin{pmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{pmatrix}.$$

   The first determinant equals the Vandermonde form

$$\det\begin{pmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{pmatrix} = (x-y)(y-z)(z-x).$$

   For the second determinant factor x, y, z from the rows respectively (or observe row-wise multiplication):

$$\det\begin{pmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{pmatrix} = xyz\det\begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix} = xyz\,(x-y)(y-z)(z-x).$$

   Combining the two parts gives the claimed result:

$$(1 + pxyz)\,(x-y)(y-z)(z-x).$$

2. Find the point on the curve $y^2 = 8x$ which is nearest to the point $(1, 4)$. [6]

   Answer: The nearest point is obtained from the parameter t satisfying

$$2t^3 + 3t - 4 = 0 \quad (\text{where } y = 4t,\ x = 2t^2).$$

   The unique real root is approximately $t \approx 0.8796$, so the nearest point is approximately

$$\big(1.5494,\ 3.5184\big).$$

(Values rounded to four decimal places.)

Solution: Parameterize the parabola $y^2 = 8x$ by

$$y = 4t, \qquad x = 2t^2 \quad (t \in \mathbb{R}).$$

Distance squared from $(2t^2, 4t)$ to $(1, 4)$ is

$$D^2(t) = (2t^2 - 1)^2 + (4t - 4)^2.$$

Differentiate and set to zero to find minima:

$$\frac{dD^2}{dt} = 2(2t^2 - 1) \cdot 4t + 2(4t - 4) \cdot 4 = 8t(2t^2 - 1) + 8(4t - 4) = 0.$$

Divide by 8:

$$t(2t^2 - 1) + (4t - 4) = 2t^3 + 3t - 4 = 0.$$

Solve for the real root numerically (the cubic has one real root relevant here):

$$t \approx 0.8796.$$

Thus

$$x = 2t^2 \approx 1.5494, \qquad y = 4t \approx 3.5184.$$

This point is the nearest point on the parabola to $(1, 4)$. (One may solve the cubic exactly by Cardano's method, but a numerical approximation is typically acceptable for the nearest-point problem.)

3. Evaluate: $\displaystyle\int \frac{dx}{3x^2 + 13x - 10}$. [6]

Answer: $\displaystyle\frac{3}{17} \ln\left|\frac{x - \frac{2}{3}}{x + 5}\right| + C.$

Solution: Factor the denominator:

$$3x^2 + 13x - 10 = 3\big(x - \tfrac{2}{3}\big)(x + 5).$$

Use partial fractions:

$$\frac{1}{3x^2 + 13x - 10} = \frac{A}{x - \frac{2}{3}} + \frac{B}{x + 5}.$$

Multiplying through gives

$$1 = A\big(x + 5\big) + B\big(x - \tfrac{2}{3}\big).$$

Setting $x = \frac{2}{3}$ yields $A = \frac{3}{17}$ and setting $x = -5$ yields $B = -\frac{3}{17}$. Therefore

$$\int \frac{dx}{3x^2 + 13x - 10} = \frac{3}{17} \int \frac{dx}{x - \frac{2}{3}} - \frac{3}{17} \int \frac{dx}{x + 5} = \frac{3}{17} \ln\left|x - \tfrac{2}{3}\right| - \frac{3}{17} \ln|x + 5| + C,$$

which simplifies to

$$\frac{3}{17} \ln\left|\frac{x - \frac{2}{3}}{x + 5}\right| + C.$$

Question 5 (15 Marks)

Answer the following questions.

(a) Prove that the relation R on the set $\mathbb{Z}$ of integers defined by xRy if $x - y$ is a multiple of 5, is an equivalence relation. [6]

Answer: R is an equivalence relation on $\mathbb{Z}$. Its equivalence classes are the congruence classes modulo 5: $[0], [1], [2], [3], [4]$.

Solution: To show R is an equivalence relation we verify reflexivity, symmetry and transitivity.

- *Reflexive:* For any $x \in \mathbb{Z}$, $x - x = 0$ which is a multiple of 5. Hence xRx for all x.
- *Symmetric:* If xRy, then $x - y = 5k$ for some integer k. Then $y - x = -5k$, which is a multiple of 5, so yRx.
- *Transitive:* If xRy and yRz, then $x - y = 5k$ and $y - z = 5\ell$ for some integers k, ℓ. Adding gives $x - z = 5(k + \ell)$, a multiple of 5, hence xRz.

Since all three properties hold, R is an equivalence relation. The equivalence classes are $\{\dots, -10, -5, 0, 5, 10, \dots\}$, $\{\dots, -9, -4, 1, 6, 11, \dots\}$, etc., i.e. the residue classes modulo 5 often denoted $[0], [1], [2], [3], [4]$.

(b) Solve the differential equation:

$$\frac{dx}{dy} + \frac{x}{y} = y^2.$$

[4]

Answer: $x(y) = \dfrac{y^3}{4} + \dfrac{C}{y}$, where C is an arbitrary constant and $y \neq 0$.

Solution: Treat x as the dependent variable and y as the independent variable. This is a linear first order equation of the form

$$\frac{dx}{dy} + \frac{1}{y}x = y^2, \qquad (y \neq 0).$$

Integrating factor:

$$\mu(y) = \exp\Big(\int \frac{1}{y}\, dy\Big) = \exp(\ln|y|) = |y|.$$

We may take $\mu(y) = y$ for $y > 0$ and similarly get the same general form; to keep formulas concise use $\mu = y$ (with the understanding $y \neq 0$). Multiply the equation by y:

$$y\frac{dx}{dy} + x = y^3 \quad \Longrightarrow \quad \frac{d}{dy}(xy) = y^3.$$

Integrate both sides:

$$xy = \frac{y^4}{4} + C \quad \Longrightarrow \quad x = \frac{y^3}{4} + \frac{C}{y},$$

which is the required general solution (defined for $y \neq 0$).

(c) A factory has two machines, Machine A and Machine B. Machine A produces 60% of the items and Machine B produces 40%. 2% of the items produced by A and 1% of the items produced by B are defective. If a defective item is chosen at random, what is the probability that it was produced by Machine B? [5]

Answer: $\dfrac{1}{4} = 0.25 = 25\%$.

Solution: Let A and B denote the events that the selected item came from Machine A or Machine B, respectively. Let D denote the event that the item is defective. Given:

$$P(A) = 0.60, \quad P(B) = 0.40, \quad P(D \mid A) = 0.02, \quad P(D \mid B) = 0.01.$$

By total probability,

$$P(D) = P(D \mid A)P(A) + P(D \mid B)P(B) = 0.02 \times 0.60 + 0.01 \times 0.40 = 0.012 + 0.004 = 0.016.$$

By Bayes' theorem,

$$P(B \mid D) = \frac{P(D \mid B)P(B)}{P(D)} = \frac{0.01 \times 0.40}{0.016} = \frac{0.004}{0.016} = \frac{1}{4} = 0.25.$$

Therefore the probability that a randomly chosen defective item was produced by Machine B is 1/4 (i.e. 25%).

SECTION B (Optional - 15 Marks)

Answer all questions from this section. (Unit V: Vectors - 5 Marks; Unit VI: 3D Geometry - 6 Marks; Unit VII: Applications of Integrals - 4 Marks)

Question 6 (5 Marks)

Answer the following questions.

1. Find the projection of the vector $\vec{a} = \hat{\imath} + 3\hat{\jmath} + 7\hat{k}$ on the vector $\vec{b} = 7\hat{\imath} - \hat{\jmath} + 8\hat{k}$. [2]
   Answer: $\mathrm{proj}_{\vec{b}}\,\vec{a} = \dfrac{60}{114}\,(7\hat{\imath} - \hat{\jmath} + 8\hat{k}) = \dfrac{10}{19}(7\hat{\imath} - \hat{\jmath} + 8\hat{k}) = \dfrac{70}{19}\,\hat{\imath} - \dfrac{10}{19}\,\hat{\jmath} + \dfrac{80}{19}\,\hat{k}.$
   Solution: Use the vector projection formula
   $$\mathrm{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\,\vec{b}.$$
   Compute the dot product and norm squared:
   $$\vec{a}\cdot\vec{b} = 1\cdot 7 + 3\cdot(-1) + 7\cdot 8 = 7 - 3 + 56 = 60, \qquad \|\vec{b}\|^2 = 7^2 + (-1)^2 + 8^2 = 49 + 1 + 64 = 114.$$
   Hence
   $$\mathrm{proj}_{\vec{b}}\,\vec{a} = \frac{60}{114}\,\vec{b} = \frac{10}{19}\,\vec{b} = \frac{10}{19}(7\hat{\imath} - \hat{\jmath} + 8\hat{k}),$$
   which gives the component form shown above.

2. Find the area of the triangle with vertices $A(1,-1,2)$, $B(2,1,-1)$ and $C(3,-1,2)$. [3]
   Answer: Area $= \sqrt{13}$.
   Solution: Let $\overrightarrow{AB} = B - A = (1,2,-3)$ and $\overrightarrow{AC} = C - A = (2,0,0)$. The area of triangle ABC is $\frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\|$.
   Compute the cross product:
   $$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 0 & 0 \end{vmatrix} = \hat{\imath}(2\cdot 0 - (-3)\cdot 0) - \hat{\jmath}(1\cdot 0 - (-3)\cdot 2) + \hat{k}(1\cdot 0 - 2\cdot 2) = (0,\,-6,\,-4).$$
   Its magnitude is $\sqrt{0^2 + (-6)^2 + (-4)^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$. Therefore the area is
   $$\frac{1}{2}\times 2\sqrt{13} = \sqrt{13}.$$

Question 7 (10 Marks)

Answer the following questions.

1. Find the distance between the parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 5 = 0$. [6]
   Answer: $\dfrac{1}{6}$.
   Solution: The normals of the two planes are proportional: the second plane's coefficients equal 2 times the first plane's coefficients. Divide the second equation by 2 to put both planes with the same normal:
   $$\text{Plane}_1:\ 2x - 2y + z + 3 = 0, \qquad \text{Plane}_2':\ 2x - 2y + z + \tfrac{5}{2} = 0.$$

The distance between two parallel planes $2x - 2y + z + c_1 = 0$ and $2x - 2y + z + c_2 = 0$ is

$$\frac{|c_2 - c_1|}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{\left|\frac{5}{2} - 3\right|}{\sqrt{4+4+1}} = \frac{\frac{1}{2}}{3} = \frac{1}{6}.$$

Thus the required distance is 1/6.

2. Using integration, find the area bounded by the parabola $y = x^2$ and the x-axis from $x = 0$ to $x = 3$. [4]

   Answer: 9.

   Solution: The area between the curve $y = x^2$ and the x-axis from $x = 0$ to $x = 3$ is

$$\int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{27}{3} - 0 = 9.$$

SECTION C (Optional - 15 Marks)

Answer all questions from this section. (Unit VIII: Application of Calculus - 5 Marks; Unit IX: Linear Regression - 6 Marks; Unit X: Linear Programming - 4 Marks)

Question 8 (5 Marks)

Answer the following question.

1. A sheet of aluminum 40 cm long and 10 cm wide is to be made into a closed box with a square base. Find the dimensions of the box that will hold maximum volume. [5]

   Answer: The box of maximum volume is a cube with square base of side

$$s = \sqrt{\frac{200}{3}} \approx 8.16497 \text{ cm},$$

and height

$$h = s \approx 8.16497 \text{ cm}.$$

Solution: Let the square base have side s and height h. A closed box (with top and bottom) has total surface area

$$S = 2s^2 + 4sh.$$

This surface area must equal the area of the sheet:

$$2s^2 + 4sh = 40 \times 10 = 400.$$

Hence

$$h = \frac{400 - 2s^2}{4s} = \frac{200 - s^2}{2s}.$$

The volume is

$$V = s^2 h = s^2 \cdot \frac{200 - s^2}{2s} = \frac{200s - s^3}{2}.$$

Differentiate V with respect to s and set to zero to find critical points:

$$\frac{dV}{ds} = \frac{200 - 3s^2}{2} = 0 \quad \Longrightarrow \quad s^2 = \frac{200}{3}.$$

Thus

$$s = \sqrt{\frac{200}{3}} \quad \text{(positive root)}.$$

Compute h from the constraint:

$$h = \frac{200 - s^2}{2s} = \frac{200 - \frac{200}{3}}{2s} = \frac{\frac{400}{3}}{2s} = \frac{200}{3s}.$$

But since $s = \sqrt{200/3}$, we get

$$h = \frac{200}{3\sqrt{200/3}} = \sqrt{\frac{200}{3}} = s.$$

So the optimal box is a cube with side $\sqrt{200/3} \approx 8.16497$ cm. A quick second-derivative check:

$$\frac{d^2V}{ds^2} = \frac{-6s}{2} = -3s < 0$$

at the critical $s > 0$, confirming a maximum.

Question 9 (10 Marks)

Answer the following questions.

1. Solve the following Linear Programming Problem graphically: Minimize $Z = 3x + 4y$ subject to

$$\begin{aligned} x + 2y &\leq 10, \\ 2x + y &\leq 10, \\ x &\geq 0, \\ y &\geq 0. \end{aligned}$$

[4]

Answer: The feasible region is the quadrilateral with vertices

$$(0,0),\ (5,0),\ \left(\tfrac{10}{3}, \tfrac{10}{3}\right),\ (0,5).$$

The minimum value of Z on the feasible region is

$$Z_{\min} = 0$$

attained at the vertex $(0,0)$. (If the trivial solution $(0,0)$ is excluded, the next smallest value is $Z = 15$ at $(5,0)$.)

Solution: Graph the two constraint lines and the coordinate axes. The intercepts are:

$$x + 2y = 10 \ \Rightarrow\ (10,0),(0,5), \qquad 2x + y = 10 \ \Rightarrow\ (5,0),(0,10).$$

With $x \geq 0, y \geq 0$ the intersection of all inequalities gives the polygon with vertices

$$(0,0),\ (5,0),\ \left(\tfrac{10}{3}, \tfrac{10}{3}\right),\ (0,5).$$

Evaluate $Z = 3x + 4y$ at these vertices:

$$\begin{aligned} (0,0):&\ \ Z = 0, \\ (5,0):&\ \ Z = 15, \\ \left(\tfrac{10}{3}, \tfrac{10}{3}\right):&\ \ Z = \frac{70}{3} \approx 23.33, \\ (0,5):&\ \ Z = 20. \end{aligned}$$

Hence the minimum is 0 at $(0,0)$. If one is seeking a nonzero production plan, the smallest nonzero value occurs at $(5,0)$ giving $Z = 15$.

2. Given $b_{xy} = 0.5$, $r = 0.7$, and $\sigma_y = 4$, find σ_x. [6]

Answer: $\sigma_x = 5.6$.

Solution: The regression coefficient of y on x (denoted b_{xy}) is related to the correlation coefficient r and standard deviations by

$$b_{xy} = r \cdot \frac{\sigma_y}{\sigma_x}.$$

Solve for σ_x:

$$\sigma_x = r \cdot \frac{\sigma_y}{b_{xy}} = 0.7 \cdot \frac{4}{0.5} = \frac{2.8}{0.5} = 5.6.$$

Thus $\sigma_x = 5.6$.