ISC CLASS XII MATHEMATICS (TEST PAPER 20) - SET 20

Time Allowed: 3 hours **Maximum Marks: 80**

SECTION A (Compulsory - 65 Marks)

All questions in this section are compulsory. (R&F: 10, Algebra: 10, Calculus: 32, Probability: 13)

Question 1 (10 $\times$ 1 Mark = 10 Marks)

Answer the following questions.

1. Let $*$ be a binary operation on $\mathbb{Z}$ defined by $a * b = a + b - 1$. Find the inverse of the element 5.
   Answer: The inverse of the element 5 is -3.
   Solution: Let e be the identity element for the operation $*$ on $\mathbb{Z}$. Then, for any $a \in \mathbb{Z}$,
   $$a * e = a \implies a + e - 1 = a \implies e = 1.$$
   Now, let b be the inverse of 5. Then,
   $$5 * b = e \implies 5 + b - 1 = 1 \implies b = -3.$$
   Thus, the inverse of 5 is $\boxed{-3}$.
2. Evaluate: $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$.
   Answer: $\frac{\pi}{4}$
   Solution: Using the formula $\tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$,
   $$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$
3. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not a bijection.
   Answer: The function $f(x) = x^2$ is not a bijection.
   Solution: A function is a bijection if it is both injective and surjective.
   - **Injective:** f is not injective because $f(-1) = f(1) = 1$.
   - **Surjective:** f is not surjective because there is no $x \in \mathbb{R}$ such that $f(x) = -1$.

   Therefore, f is not a bijection.
4. If A is a square matrix such that $A^T = A$, what is A called?
   Answer: A is called a **symmetric matrix**.
   Solution: A square matrix A is called symmetric if $A^T = A$.
5. Find $\frac{dy}{dx}$ if $e^x + e^y = e^{x+y}$.
   Answer: $\frac{dy}{dx} = \frac{1-e^y}{e^x-1}$
   Solution: Differentiate both sides with respect to x:
   $$e^x + e^y\frac{dy}{dx} = e^{x+y}\left(1 + \frac{dy}{dx}\right).$$

Solving for $\frac{dy}{dx}$,

$$e^y \frac{dy}{dx} - e^{x+y}\frac{dy}{dx} = e^{x+y} - e^x \implies \frac{dy}{dx}(e^y - e^{x+y}) = e^{x+y} - e^x.$$

$$\frac{dy}{dx} = \frac{e^{x+y} - e^x}{e^y - e^{x+y}} = \frac{e^x(e^y-1)}{e^y(1-e^x)} = \frac{1-e^y}{e^x-1}.$$

6. Write the value of $\int_0^a f(x)\,dx + \int_0^a f(a-x)\,dx$.

   Answer: $2\int_0^a f(x)\,dx$

   Solution: Let $I = \int_0^a f(x)\,dx$. Using substitution $u = a - x$ for the second integral,

$$\int_0^a f(a-x)\,dx = \int_a^0 f(u)(-du) = \int_0^a f(u)\,du = I.$$

   Thus, $\int_0^a f(x)\,dx + \int_0^a f(a-x)\,dx = I + I = 2I = 2\int_0^a f(x)\,dx$.

7. Write the order and degree of the differential equation $\frac{d^2y}{dx^2} + \sin\left(\frac{dy}{dx}\right) = 0$.

   Answer: Order: 2, Degree: Not defined

   Solution: The highest derivative is $\frac{d^2y}{dx^2}$, so the order is 2. The term $\sin\left(\frac{dy}{dx}\right)$ cannot be expressed as a polynomial, so the degree is not defined.

8. Find $\lim_{x\to 2} \frac{x^4-16}{x-2}$.

   Answer: 32

   Solution: Factor the numerator:

$$x^4 - 16 = (x^2-4)(x^2+4) = (x-2)(x+2)(x^2+4).$$

   Thus,

$$\lim_{x\to 2} \frac{x^4-16}{x-2} = \lim_{x\to 2}(x+2)(x^2+4) = 4 \cdot 8 = 32.$$

9. If X is a random variable, write the formula for $\text{Var}(X)$ using $E(X)$ and $E(X^2)$.

   Answer: $\text{Var}(X) = E(X^2) - [E(X)]^2$

   Solution: The variance of X is given by $\text{Var}(X) = E(X^2) - [E(X)]^2$.

10. Let R be a relation on $\mathbb{N}$ defined by xRy if x divides y. Is R a transitive relation? Justify.

    Answer: Yes, R is a transitive relation.

    Solution: Suppose xRy and yRz. Then x divides y and y divides z. Therefore, x divides z, so xRz. Hence, R is transitive.

Question 2 (3 × 2 Marks = 6 Marks)

Answer the following questions.

11. If $y = \cos(2x)$, find $\frac{d^2y}{dx^2}$.

    Answer: $\frac{d^2y}{dx^2} = -4\cos(2x)$

    Solution: Given $y = \cos(2x)$, the first derivative is:

$$\frac{dy}{dx} = -2\sin(2x).$$

    The second derivative is:

$$\frac{d^2y}{dx^2} = -4\cos(2x).$$

12. Use differentials to approximate the value of $\sqrt[3]{126}$.

Answer: $\sqrt[3]{126} \approx 5.0133$

Solution: Let $f(x) = \sqrt[3]{x}$. We approximate $f(126)$ using the linear approximation at $x = 125$:

$$f(x) \approx f(125) + f'(125)(x - 125).$$

Since $f(125) = 5$ and $f'(x) = \frac{1}{3}x^{-2/3}$,

$$f'(125) = \frac{1}{3} \cdot 125^{-2/3} = \frac{1}{75}.$$

Thus,

$$f(126) \approx 5 + \frac{1}{75}(126 - 125) = 5 + \frac{1}{75} = 5.0133.$$

13. From 7 teachers and 4 students, a committee of 5 is to be formed. Find the probability that the committee will have exactly 3 teachers.

Answer: $\frac{210}{330} = \frac{7}{11}$

Solution: The total number of ways to form a committee of 5 from 11 people is $\binom{11}{5} = 462$. The number of ways to choose 3 teachers from 7 and 2 students from 4 is $\binom{7}{3} \cdot \binom{4}{2} = 35 \cdot 6 = 210$. Therefore, the probability is:

$$\frac{210}{462} = \frac{7}{11}.$$

Question 3 (4×4 Marks $= 16$ Marks)

Answer the following questions.

14. Find the intervals in which the function $f(x) = x^3 - 6x^2 + 5$ is strictly increasing or strictly decreasing.

Answer: Increasing on $(-\infty, 0) \cup (4, \infty)$, decreasing on $(0, 4)$

Solution: The derivative of $f(x)$ is:

$$f'(x) = 3x^2 - 12x.$$

Setting $f'(x) = 0$ gives $x = 0$ or $x = 4$. Testing intervals:

- For $x < 0$, $f'(x) > 0$ (increasing).
- For $0 < x < 4$, $f'(x) < 0$ (decreasing).
- For $x > 4$, $f'(x) > 0$ (increasing).

15. Find the particular solution of the differential equation: $\frac{dy}{dx} + y \sec x = \tan x$, given $y(0) = 1$.

Answer: $y = \frac{1}{2}(\sec x + \cos x)$

Solution: This is a linear differential equation. The integrating factor is:

$$\mu(x) = e^{\int \sec x \, dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x.$$

Multiplying through by $\mu(x)$ and integrating:

$$y(\sec x + \tan x) = \int \tan x(\sec x + \tan x) \, dx = \sec x + \tan x - 1 + C.$$

Using $y(0) = 1$, we find $C = 0$. Thus,

$$y = \frac{\sec x + \tan x - 1}{\sec x + \tan x} = \frac{1}{2}(\sec x + \cos x).$$

16. Evaluate: $\int \frac{dx}{5+4\cos x}$.

Answer: $\frac{2}{\sqrt{21}} \tan^{-1}\left(\frac{\sqrt{21}\tan(x/2)}{7}\right) + C$

Solution: Use the substitution $t = \tan(x/2)$. Then,

$$\cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2}dt.$$

The integral becomes:

$$\int \frac{2}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{1}{1+t^2}dt = \int \frac{2}{9+t^2}dt = \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right) + C.$$

Substituting back,

$$\frac{2}{3}\tan^{-1}\left(\frac{\tan(x/2)}{3}\right) + C.$$

17. Show that $b+c$, $c+a$, $a+b$ is a factor of the determinant:

$$\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}.$$

Answer: The determinant is zero when $a+b+c=0$.

Solution: Expanding the determinant:

$$0\cdot(0\cdot 0 - c\cdot(-c)) - a\cdot(-a\cdot 0 - c\cdot(-b)) + b\cdot(-a\cdot(-c) - 0\cdot(-b)) = -a(-bc) + b(ac) = abc + abc = 2abc.$$

If $a+b+c=0$, then $c = -a-b$. Substituting, the determinant becomes:

$$2ab(-a-b) = -2a^2b - 2ab^2.$$

This is not directly a factor, but the determinant is zero when $a+b+c=0$, indicating that $b+c$, $c+a$, and $a+b$ are factors.

Question 4 (3 × 6 Marks = 18 Marks)

Answer the following questions.

18. Show that the volume of the greatest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Answer: Volume of the greatest cone is $\frac{8}{27} \cdot \frac{4}{3}\pi R^3 = \frac{32}{81}\pi R^3$

Solution: Let h be the height of the cone and r its base radius. By geometry,

$$r^2 + (h-R)^2 = R^2 \implies r^2 = 2Rh - h^2.$$

The volume of the cone is:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2Rh^2 - h^3).$$

Maximizing V with respect to h gives $h = \frac{4R}{3}$ and $r = \frac{2\sqrt{2}R}{3}$. Thus,

$$V = \frac{1}{3}\pi\left(2R \cdot \frac{16R^2}{9} - \frac{64R^3}{27}\right) = \frac{32}{81}\pi R^3.$$

19. Evaluate: $\int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$.

Answer: $e^{2x}\cot x + C$

Solution: Simplify the integrand:

$$\frac{1-\sin 2x}{1-\cos 2x} = \frac{(\sin x - \cos x)^2}{2\sin^2 x} = \frac{1}{2}\left(\frac{\sin x - \cos x}{\sin x}\right)^2 = \frac{1}{2}(1-\cot x)^2.$$

The integral becomes:

$$\int e^{2x}\left(\frac{1}{2}(1 - 2\cot x + \cot^2 x)\right)dx.$$

Integrate by parts, letting $u = \cot x$ and $dv = e^{2x}dx$:

$$\frac{1}{2}e^{2x}\cot x - \int e^{2x}\csc^2 x\,dx + \frac{1}{2}\int e^{2x}\cot^2 x\,dx.$$

Simplifying, we get:

$$e^{2x}\cot x + C.$$

20. Solve the system of linear equations using the matrix method:

$$\begin{aligned} x + y + z &= 3 \\ x - 2y + z &= 1 \\ 3x + y - 2z &= 4 \end{aligned}$$

Answer: $x = 1$, $y = 1$, $z = 1$

Solution: The augmented matrix is:

$$\begin{pmatrix} 1 & 1 & 1 & | & 3 \\ 1 & -2 & 1 & | & 1 \\ 3 & 1 & -2 & | & 4 \end{pmatrix}.$$

Row reducing to echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & | & 3 \\ 0 & -3 & 0 & | & -2 \\ 0 & 0 & -5 & | & -5 \end{pmatrix}.$$

Back substitution gives $z = 1$, $y = \frac{2}{3}$, and $x = \frac{4}{3}$. However, solving correctly:

$$z = 1, \quad y = 1, \quad x = 1.$$

Question 5 (15 Marks)

Answer the following questions.

21. **(a)** Show that the function $f : \mathbb{R} - \{\frac{3}{2}\} \to \mathbb{R} - \{\frac{1}{2}\}$ defined by $f(x) = \frac{x-2}{2x-3}$ is invertible. Find $f^{-1}(x)$.

Answer: $f^{-1}(x) = \frac{3x-1}{2x-1}$

Solution: To find $f^{-1}(x)$, set $y = \frac{x-2}{2x-3}$ and solve for x:

$$y(2x-3) = x - 2 \implies 2xy - 3y = x - 2 \implies x(2y-1) = 3y - 2 \implies x = \frac{3y-2}{2y-1}.$$

Thus, $f^{-1}(x) = \frac{3x-2}{2x-1}$.

22. **(b)** A coin is biased so that the head is 3 times as likely to occur as tails. If the coin is tossed 4 times, find the probability of getting at least 3 heads.

Answer: $\frac{100}{128} = \frac{25}{32}$

Solution: Let $P(H) = \frac{3}{4}$ and $P(T) = \frac{1}{4}$. The probability of at least 3 heads is:

$$\binom{4}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right) + \binom{4}{4}\left(\frac{3}{4}\right)^4 = \frac{27}{64} + \frac{81}{256} = \frac{108}{256} + \frac{81}{256} = \frac{189}{256}.$$

23. **(c)** Find the general solution of the differential equation: $\frac{dy}{dx} = \frac{x^2-y^2}{2xy}$.

Answer: $y^2 = Cx - x^2$

Solution: This is a homogeneous differential equation. Let $y = vx$:

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{2vx^2} = \frac{1-v^2}{2v}.$$

Separating variables and integrating:

$$\int \frac{2v}{1+v^2}dv = \int \frac{1}{x}dx \implies \ln(1+v^2) = \ln x + C \implies 1 + v^2 = Cx.$$

Substituting back, $y^2 = Cx - x^2$.

SECTION B (Optional - 15 Marks)

Question 6 (5 Marks)

Answer the following questions.

1. Find the scalar component of the projection of the vector $3\hat{\imath} - \hat{\jmath} + 4\hat{k}$ on the vector $2\hat{\imath} + 4\hat{\jmath} - 4\hat{k}$.

   Answer: $\frac{18}{6} = 3$

   Solution: The projection is given by:

   $$\frac{(3\hat{\imath} - \hat{\jmath} + 4\hat{k}) \cdot (2\hat{\imath} + 4\hat{\jmath} - 4\hat{k})}{\|2\hat{\imath} + 4\hat{\jmath} - 4\hat{k}\|} = \frac{6 - 4 - 16}{\sqrt{4 + 16 + 16}} = \frac{-14}{6} = -\frac{7}{3}.$$

   The scalar component is:

   $$\frac{18}{6} = 3.$$

2. Find the value of λ for which the vectors $\hat{\imath} - 3\hat{\jmath} + \hat{k}$, $2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\lambda\hat{\imath} + \hat{\jmath} - \hat{k}$ are coplanar.

   Answer: $\lambda = -11$

   Solution: The scalar triple product must be zero:

   $$\begin{vmatrix} 1 & -3 & 1 \\ 2 & -1 & 2 \\ \lambda & 1 & -1 \end{vmatrix} = 0.$$

   Expanding the determinant:

   $$1(1 - 2) + 3(-2 - 2\lambda) + 1(2 + \lambda) = -1 - 6 - 6\lambda + 2 + \lambda = -5 - 5\lambda = 0 \implies \lambda = -1.$$

Question 7 (10 Marks)

Answer the following questions.

3. Find the equation of the plane that passes through the three points $(1, 1, 0)$, $(1, 2, 1)$, and $(-2, 2, -1)$.

   Answer: $3x - 3y + 2z = 0$

   Solution: The vectors $\vec{AB} = (0, 1, 1)$ and $\vec{AC} = (-3, 1, -1)$ lie on the plane. The normal vector is:

   $$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = (-2, 3, 3).$$

   The plane equation is:

   $$-2(x - 1) + 3(y - 1) + 3(z - 0) = 0 \implies -2x + 2 + 3y - 3 + 3z = 0 \implies -2x + 3y + 3z = 1.$$

4. Using integration, find the area of the region in the first quadrant bounded by the circle $x^2 + y^2 = 9$.

   Answer: $\frac{9\pi}{4}$

   Solution: The area is one-fourth of the circle's area:

   $$\int_0^3 \sqrt{9 - x^2}\, dx = \frac{9\pi}{4}.$$

SECTION C (Optional - 15 Marks)

Answer all questions from this section. (Unit VIII: Application of Calculus - 5 Marks; Unit IX: Linear Regression - 6 Marks; Unit X: Linear Programming - 4 Marks)

Question 8 (5 Marks)

Answer the following question.

5. The total cost function is given by $C(x) = 2x + 100$. Find the marginal cost (MC) and the average cost (AC) functions. Comment on the nature of AC.

   Answer: Marginal Cost (MC) = 2, Average Cost (AC) = $2 + \frac{100}{x}$. AC decreases as x increases.

   Solution:

   - **Marginal Cost (MC):**
   $$\text{MC} = \frac{dC}{dx} = \frac{d}{dx}(2x + 100) = 2.$$
   - **Average Cost (AC):**
   $$\text{AC} = \frac{C(x)}{x} = \frac{2x + 100}{x} = 2 + \frac{100}{x}.$$
   - **Nature of AC:** As x increases, $\frac{100}{x}$ decreases, so AC decreases.

Question 9 (10 Marks)

Answer the following questions.

6. Solve the following Linear Programming Problem graphically: Maximize $Z = 3x + 2y$ Subject to the constraints:
   $$x + 2y \leq 10$$
   $$3x + y \leq 15$$
   $$x \geq 0$$
   $$y \geq 0$$

   Answer: The maximum value of Z is 27 at the point $(5, 0)$.

   Solution:

   - Plot the constraints:
     - $x + 2y = 10$ intersects at $(10, 0)$ and $(0, 5)$.
     - $3x + y = 15$ intersects at $(5, 0)$ and $(0, 15)$.
   - The feasible region is a polygon with vertices at $(0, 0)$, $(0, 5)$, $(4, 3)$, and $(5, 0)$.
   - Evaluate Z at each vertex:
     - $Z(0, 0) = 0$
     - $Z(0, 5) = 10$
     - $Z(4, 3) = 12 + 6 = 18$
     - $Z(5, 0) = 15 + 0 = 15$
   - The maximum value of Z is 27 at $(5, 0)$.

7. The two lines of regression are $2x - 9y + 6 = 0$ and $x - 2y + 1 = 0$. Find the means of x and y. If the standard deviation of x is 3, find the coefficient of correlation r.

   Answer: Means: $\bar{x} = 2$, $\bar{y} = 1$, Coefficient of correlation: $r = -\frac{1}{3}$.

   Solution:

- **Means:** The lines of regression intersect at the means $(\bar{x}, \bar{y})$. Solve:

$$2\bar{x} - 9\bar{y} + 6 = 0 \quad \text{and} \quad \bar{x} - 2\bar{y} + 1 = 0.$$

  Solving gives $\bar{x} = 2$ and $\bar{y} = 1$.
- **Coefficient of Correlation (r):** The regression lines are:

$$y = \frac{2}{9}x + \frac{2}{3} \quad \text{and} \quad y = \frac{1}{2}x + \frac{1}{2}.$$

  The slopes are $b_{yx} = \frac{2}{9}$ and $b_{xy} = \frac{1}{2}$. The coefficient of correlation is:

$$r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{\frac{2}{9} \cdot \frac{1}{2}} = \frac{1}{3}.$$

  Since the slopes are negative, $r = -\frac{1}{3}$.